Circulation of groundwater due to wave set-up on a permeable beach

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Abstract

Sandy beaches are highly exploited but very dynamic and fragile environments. Driven by waves, the water flow through the beach body is able to transport oxygen, and hence help to maintain biological activity in the porous media. The paper presents a theoretical attempt to predict the groundwater circulation due to wave set-up. Two systems of circulations have been discovered, related to two different gradients of the set-up height. For the offshore gradient, the horizontal excess pressure gradient induces flow in the offshore direction. However, closer to the shore, the pressure gradient is reversed and the resulting flow moves shorewards.

1. Introduction

In a recent paper, which will be referred to as paper I, Massel & Pelinovsky (2001) considered the run-up of dispersive and breaking waves on a gentle beach slope. It was shown that in the set-up this phenomenon has a considerable effect on the resulting run-up height. The set-up mechanism is of the phase-averaged type and results from the balance between the radiation stress S_{xx} (Longuet-Higgins & Stewart 1964) and the denivelation relative to the still water level $\bar{\zeta}(x)$, i.e.

$$\frac{dS_{xx}}{dx} + \rho g(h + \bar{\zeta}(x))\frac{d\bar{\zeta}(x)}{dx} = 0, \tag{1}$$

in which ρ is the water density and h is the water depth relative to the still water level.

However, in paper I the permeability of the sea bottom was neglected. When the sea bottom is permeable, the denivelation of the mean sea level $\bar{\zeta}$ induces a groundwater circulation which contributes to the submarine groundwater discharge (Longuet-Higgins 1983, Kang & Nielsen 1996, Li & Barry 2000). This situation is shown diagrammatically in Fig. 1 (see Region 3 between points $\mathrm{B_b}$ and D). The velocity of flow as well as the amount of water circulating within the permeable beach body is important for the biological status of the organisms inhabiting the beach sand. Although the biodiversity and biomass of interstitial organisms within the beach body are low, recent findings have shown that marine sands transfer energy very effectively (Węsławski et al. 2000). Moreover, the chemical and biological reactions are faster than in the fine-grained sediments. The high diversity of diatoms and meiofauna in undisturbed beaches may act as effective biological filters for some types of pollutants, and more abundant biota in disturbed beaches are more effective in processing organic matter and beach self-cleaning. Some pollutants, such as hydrocarbons and heavy metals, are sorbed on the surface of microbes and diatoms (Siron et al. 1996). Another important function of sandy beaches is the decomposition of dead organic matter. For the Baltic range this is 1 to 20 g fresh weight per m^2 per day reduced to mineral elements (Węsławski et al. 2000).

Little is known about the water flow in this region. Longuet-Higgins (1983) developed a simple analytical solution for the circulation induced by wave set-up. The problem was solved for a semi-infinite domain, but the free surface boundary conditions at the water table and the landward boundaries were not included in the solution. Li & Barry (2000) presented a numerical study of the instantaneous, phase-resolved wave motion and resulting groundwater variation in the beach zone due to progressive bore. They also considered the averaged flow due to wave set-up using a simplified representation of the set-up gradient.

In this short paper, a simplified model of the wave-bottom interaction in the set-up region is presented. The simplification is based on the assumption that the phase-averaged, mean pressure gradient, though small, produces effects that, because they are cumulative in time, may be more far-reaching. Special attention is given to the determination of the kinematic characteristics of flow and their dependence on the incident wave parameters. The paper is organized as follows. The simplified wave set-up model is briefly discussed in Section 2. The fundamental governing equations for the water circulation due to set-up are developed in Section 3 and the solution of the governing equations is discussed in Section 4. Section 5 contains examples of numerical calculations. Some remarks on the layout of possible experiments and main conclusions are listed in Section 6.

2. Simplified model for the wave set-up

As was mentioned in Section 1, the gradient of the radiation stress tensor S_{xx} induces a denivelation of the mean water level $\bar{\zeta}$. Before the breaking point, the wave height changes a little and the resulting wave set-up is very small. However, from the breaking point, the wave set-up increases substantially. To calculate the set-up height $\bar{\zeta}(x)$ in Region 3, we use the formula resulting from the shallow-water approximation of eq. (1) given in paper I

$$\bar{\zeta}(x) = \bar{\zeta}_{br} + \frac{3}{8}\gamma_{br}^2 \left(1 + \frac{3}{8}\gamma_{br}^2\right)^{-1} \left[h_{br} - h(x)\right], \tag{2}$$

in which $\gamma_{br} = (\frac{H}{h})_{br}$, $\bar{\zeta}_{br} = \frac{1}{16}\gamma_{br} H_{br}$ is the set-down value at the breaking point and H_{br} is the breaking wave height. We note that for the still-water level $(x = 0)$ eq. (2) yields

$$\bar{\zeta}_0 = \bar{\zeta}(0) = \bar{\zeta}_{br} + \frac{3}{8}\gamma_{br}^2 \left(1 + \frac{3}{8}\gamma_{br}^2\right)^{-1} h_{br}. \tag{3}$$

Eq. (2) indicates that when water depth $h(x) = \beta x$ (β is the bottom slope), the set-up height becomes a linear function of x.

3. Governing equations for groundwater circulation due to wave set-up

Beaches consisting of sand or unconsolidated sediment are porous and any changes of pressure associated with the wave set-up produce a flow of sea water within the beach itself. The oscillatory component of the pressure may produce some damping of the waves over a porous beach (Massel 1976). On the other hand, the wave motion percolating through a permeable bottom influences the wave forces on the hydraulic structures supported by or extending into the bottom. These wave forces are able to effect the behaviour of a porous material underneath and in the vicinity of a structure.

Let us assume that the sand is anisotropic and the flow is two-dimensional in the plane $(0, x, z)$. The flow is considered to be in the Darcy law range, while the soil is assumed to be fully saturated (no air is contained in the porous media) and the grain skeleton is rigid. The complete equations of motion of the soil element are (Moshagen & Torum 1975)

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} &= -\frac{1}{n\varrho}\frac{\partial p}{\partial x} - \frac{g}{nK}\,u \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} &= -\frac{1}{n\varrho}\frac{\partial p}{\partial z} - \frac{g}{nK}\,w \end{aligned}\right\}, \tag{4}$$

and the equation of continuity is

$$\frac{u}{\varrho}\frac{\partial \varrho}{\partial x} + \frac{w}{\varrho}\frac{\partial \varrho}{\partial z} + \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = -\frac{n}{E}\frac{\partial p}{\partial t}, \tag{5}$$

in which n is the ratio of pore volume to total volume, x, y are the horizontal and vertical coordinates (see Fig. 1), K is the coefficient of permeability, p is the excess pressure of water, u and w are the Darcy velocities in the x and z directions, respectively, and E is the bulk modulus of the water $= 2.3 \times 10^9$ N m^{-2}.

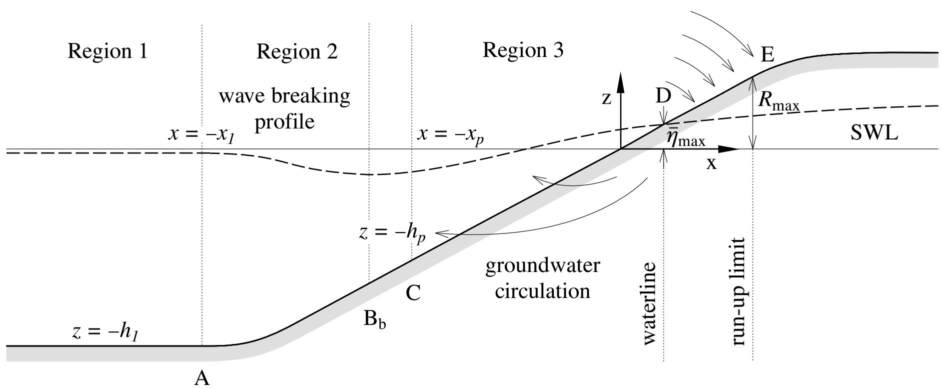

Fig. 1. Reference scheme

We estimate the coefficient of permeability K using the approximate Hazen formula

$$K = (1.0\text{–}1.5)\, D_{10}^2, \tag{6}$$

in which K is in meters per second when D_{10} (grain diameter) is in centimeters.

To get the solution in closed form, we simplify the nonlinear equations (4)–(5). Using the fact that the nonlinear terms in these equations are negligible for slow motion and that for a stationary, phase-averaged flow the local accelerations are zero, we get

$$\left.\begin{aligned} u &= -\frac{K}{\gamma}\frac{\partial p}{\partial x} \\ w &= -\frac{K}{\gamma}\frac{\partial p}{\partial z} \end{aligned}\right\}, \tag{7}$$

where γ is the unit weight of water, and

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{8}$$

Therefore, the equation for the pressure response becomes the Laplace equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} = 0. \tag{9}$$

Let us introduce a new coordinate system (x_1, z_1) – Fig. 2. The coordinates of points in this system and the initial one (x, z) are related as follows:

$$\left.\begin{aligned} x_1 &= (x + x_0)\cos\theta_{sl} + (z + h_0)\sin\theta_{sl} - \frac{l}{2} \\ z_1 &= -(x + x_0)\sin\theta_{sl} + (z + h_0)\cos\theta_{sl} \end{aligned}\right\}, \tag{10}$$

where θ_{sl} is the angle of beach slope, and x_0, h_0 and l are defined in Fig. 2.

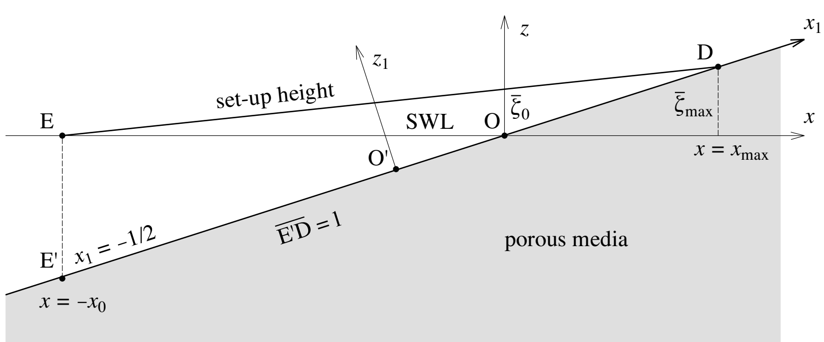

Fig. 2. Local coordinate systems

It can be shown that the Laplace equation (9) is invariant against the transformation (10); hence, the equation for excess pressure in a new coordinate system becomes

$$\frac{\partial^2 p}{\partial x_1^2} + \frac{\partial^2 p}{\partial z_1^2} = 0. \tag{11}$$

To define the boundary conditions at the sea bottom between points E′ and D (see Fig. 2), we assume approximately that the set-down height $\bar{\zeta}_{br}$ is negligibly small. Therefore, the excess pressure loading at the sea bottom (i.e. along the axis $z_1 = 0$) takes the form

$$p_0(x_1) = p(x_1, 0) = \begin{cases} 0 & \text{for} \quad x_1 \leq -\frac{l}{2}, \\ p_0^{(\max)} \left(x_1 + \frac{l}{2}\right) & \text{for} \quad -\frac{l}{2} \leq x_1 \leq \frac{l_1 - l_2}{2}, \\ p_0^{(\max)} \left[1 - \frac{1}{l_2}\left(x - \frac{l_1 - l_2}{2}\right)\right] & \text{for} \quad \frac{l_1 - l_2}{2} \leq x_1 \leq \frac{l}{2}, \\ 0 & \text{for} \quad x_1 > \frac{l}{2}, \end{cases} \tag{12}$$

where

$$p_0^{(\max)} = \rho g \bar{\zeta}_0, \tag{13}$$

and $l = \overline{E'D}$, $l_1 = \overline{E'O}$, $l_2 = \overline{OD}$.

For the infinite thickness of the porous layer we assume

$$p \longrightarrow 0 \quad \text{when} \quad z_1 \longrightarrow -\infty. \tag{14}$$

This means that the domain of motion is semi-infinite and the effects of the acquifer's bottom and landward boundaries were not included in the solution. Summarizing the above formulation, we are looking for the solution of the Laplace equation (11) with the boundary conditions (12) and (14) in the coordinate system (x_1, z_1).

4. Solution of governing equations

To solve the boundary value problem defined in Section 3, we apply an approach similar to that suggested by Longuet-Higgins (1983). Therefore, let us define the elliptical coordinate system (η, θ) as follows:

$$\left.\begin{aligned} x_1 &= \tfrac{l}{2} \cosh \eta \cos \theta \\ z_1 &= \tfrac{l}{2} \sinh \eta \sin \theta \end{aligned}\right\}, \tag{15}$$

for $0 \leq \eta < \infty$ and $\pi \leq \theta \leq 2\pi$.

The Laplace equation (11) for the excess pressure p in the elliptical system is (Moon & Spencer 1961)

$$\frac{4}{l^2(\cosh^2 \eta - \cos^2 \theta)} \left(\frac{\partial^2 p}{\partial \eta^2} + \frac{\partial^2 p}{\partial \theta^2}\right) = 0 \quad \text{or} \quad \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial \theta^2} = 0 \tag{16}$$

and the boundary condition at the sea bottom ($\eta = 0$) becomes

$$p_0(\theta) = \begin{cases} p_0^{(\max)}\, a\,(1+\cos\theta) & \text{for} \quad \pi \le \theta \le \theta_0 = 2\pi - \arccos\left(\frac{l_1 - l_2}{l}\right), \\ p_0^{(\max)}\, b\,(1-\cos\theta) & \text{for} \quad \theta_0 \le \theta \le 2\pi \end{cases} \tag{17}$$

in which

$$a = \frac{l}{2l_1}, \qquad b = \frac{l}{2l_2}. \tag{18}$$

In general, the set-up $\bar{\zeta}(x)$ is not a linear function of x. Therefore, in order to develop a more generic solution we present the pressure $p(\theta)$ in the form of a Fourier series:

$$p_0(\theta) = p_0^{(\max)} \sum_{n=0}^{\infty} \left[a_n\, \cos(2n\theta) + b_n\, \sin(2n\theta)\right], \tag{19}$$

where

$$a_0 = \frac{1}{\pi}[(2b-a)\pi + (a-b)\theta_0 + (a+b)\sin\theta_0], \quad b_0 = 0, \tag{20}$$

$$a_n = \frac{2}{\pi}\left[\frac{(a-b)\,\sin(2n\theta_0)}{2n} + \frac{(a+b)\,\sin(2n-1)\theta_0}{2(2n-1)} + \frac{(a+b)\,\sin(2n+1)\theta_0}{2(2n+1)}\right], \tag{21}$$

$$b_n = \frac{2}{\pi}\left\{\frac{(a-b)[1-\cos(2n\theta_0)]}{2n} + \frac{(b-a)-(a+b)\,\cos(2n+1)\theta}{2(2n+1)} + \frac{(b-a)-(a+b)\,\cos(2n-1)\theta}{2(2n-1)}\right\}. \tag{22}$$

We are seeking the solution of eq. (16) in the form

$$p(\eta,\theta) = p_0^{(max)} \sum_{n=0}^{\infty} e^{-2n\eta}[a_n\, \cos(2n\theta) + b_n\, \sin(2n\theta)]. \tag{23}$$

It should be noted that eq. (23) satisfies the Laplace equation (16), the boundary condition at the sea bottom. Very far away from the sea bottom the solution is

$$p(\eta,\theta) \longrightarrow 0 \quad \text{for} \quad \eta \longrightarrow \infty. \tag{24}$$

Now using formulas (7) we can determine the velocity of groundwater percolating through the beach itself in the form

$$\vec{u} = -\frac{K}{\gamma}\, grad\,(p). \tag{25}$$

In the elliptical coordinate system (η, θ), eq. (25) becomes (Moon & Spencer 1961)

$$\vec{u}(\eta,\theta) = u_\eta\,\vec{a}_\eta + u_\theta\,\vec{a}_\theta = -\frac{K}{\gamma}\,\frac{2}{l\,\sqrt{\cosh^2\eta - \cos^2\theta}}\left(\frac{\partial p}{\partial \eta}\,\vec{a}_\eta + \frac{\partial p}{\partial \theta}\,\vec{a}_\theta\right), \qquad (26)$$

where $\vec{a}_\eta$ and $\vec{a}_\theta$ are the unit vectors in the directions η and θ, respectively. After differentiating the excess pressure with respect to η and θ, the corresponding velocity components become

$$u_\eta = \frac{4p_0^{(\max)}K}{l\gamma\sqrt{\cosh^2\eta - \cos^2\theta}}\sum_{n=0}^{\infty} n\,e^{-2n\eta}[a_n\cos(2n\theta) + b_n\sin(2n\theta)], \quad (27)$$

$$u_\theta = \frac{4p_0^{(\max)}K}{l\gamma\sqrt{\cosh^2\eta - \cos^2\theta}}\sum_{n=0}^{\infty} n\,e^{-2n\eta}[a_n\sin(2n\theta) - b_n\cos(2n\theta)]. \quad (28)$$

Hence, the module of the velocity is

$$\mid u(\eta,\theta) \mid = \sqrt{u_\eta^2 + u_\theta^2}. \qquad (29)$$

For a better illustration of the circulation pattern, the stream function, which by definition is constant along the streamlines of the flow, is very useful. It can be found that the stream function $\psi(\eta,\theta)$ for groundwater flow is

$$\psi(\eta,\theta) = \psi_0\sum_{n=0}^{\infty} e^{-2n\eta}[b_n\ \cos(2n\theta) - a_n\ \sin(2n\theta)], \qquad (30)$$

where

$$\psi_0 = -p_0^{(\max)}\frac{K}{\gamma}. \qquad (31)$$

5. Examples of calculations

In order to demonstrate the application of the preceding formulas, we calculate the flow of the groundwater within the beach body for two different scenarios. For both cases the wave set-up characteristics have been calculated by using the method given in paper I.

The first scenario deals with the beach slope $\beta = 0.1$ and sand characterized by diameter $D_{10} = 0.2$ mm. The incident wave period is $T = 6$ s and the deep-water wave height $H_0 = 4.38$ m. Using the experimental arrangements reported by Saville (1958) we found $\gamma_{br} = (\frac{H}{h})_{br} = 0.91$ and $\bar{\zeta}_{\max} = 0.92$ m.

The other distances and values defined in Fig. 2 become $x_0 = 38.75$ m, $\zeta_0 = 0.91$ m, $l_1 = 38.94$ m, $l_2 = 9.25$ m and $l = 48.19$ m.

Fig. 3 shows the resulting stream function $\psi(\eta, \theta)$. The flow extends considerably beyond the segment between points E′ and D, where the external excess pressure is applied. Two systems of the groundwater circulation related to different excess pressure gradients (see eq. (12)) can be clearly distinguished. Due to the positive horizontal gradient of the excess pressure associated with the wave set-up in the segment between points E′ and O, the flow is in the offshore direction – see arrows in Fig. 3. This means that the pressure gradient is sufficiently strong to swamp the viscous forces in the laminar boundary layer. This is in contrast to the typical situation of a uniform progressive wave travelling over a horizontal bottom when at the bottom water particles tend to move forwards in the direction of wave propagation.

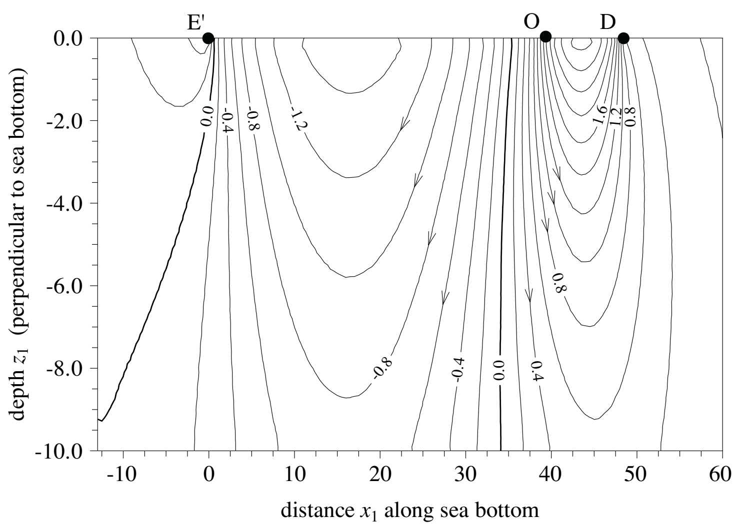

Fig. 3. Streamline pattern for an incident wave period $T = 6$ s and beach slope $\beta = 0.1$

The second system of circulation is induced by the negative pressure gradient applied in the segment between points O and D. This gradient induces the flow to move somewhat shorewards. Demarcation lines between the two systems are indicated in Fig. 3 by the stream function values equal to zero.

The second example deals with the set-up of waves of period $T = 8$ s on a steeper beach, $\beta = 0.167$, also considered in Saville's (1958) experiments. The set-up geometry is characterized by the following values: $l_1 = 41.07$ m,

$l_2 = 11.05$ m, $l = 52.12$ m, $x_0 = 40.51$, $\bar{\zeta}_{max} = 1.82$ and $\bar{\xi}_0 = 1.81$ m. The pattern of the streamlines is shown in Fig. 4. As the external excess pressure is larger than in the previous case, the resulting stream functions are higher. However, the layout of streamlines is very similar to that given in Fig. 3.

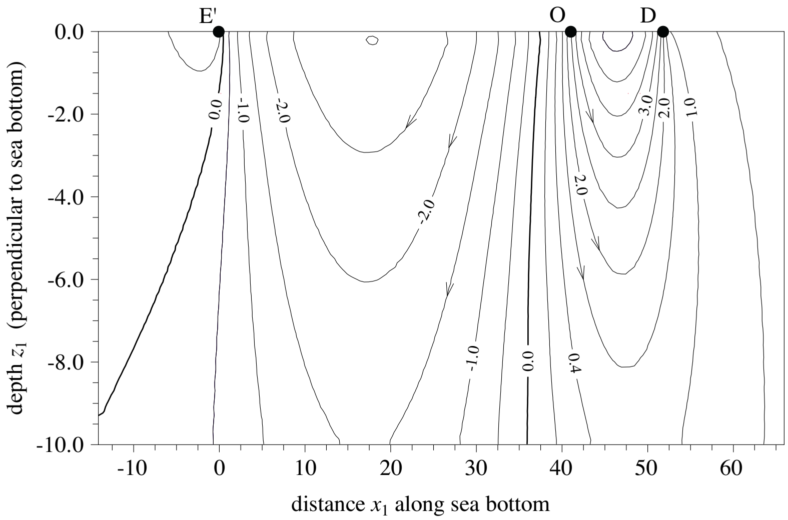

Fig. 4. Streamline pattern for an incident wave period $T = 8$ s and beach slope $\beta = 0.167$

It should be noted that in both cases the infiltration starts close to the point O where the external excess pressure reaches its maximum value. The exfiltration is located on the lower part of the beach for the first system and near the waterline for the second one. In both figures, the second system is much smaller than the first system. Moreover, it should be noted that the obtained pattern of streamlines is not symmetrical with respect to the local z_1 axis, unlike the Longuet-Higgins (1983) solution. This difference is due to the more complex excess pressure loading used in our solution.

In order to estimate the velocity of water percolating through the beach, the transect for $\eta = 0.01$ was considered. This value corresponds to the upper bottom layer of the thickness of about 2 cm. In Fig. 5, the normalized velocity component

$$\frac{l\gamma\sqrt{\cosh^2\eta - \cos^2\theta}}{4p_0^{(\max)}K}\, u_\theta = \sum_{n=0}^{\infty} n\, e^{-2n\eta}[a_n \sin(2n\theta) - b_n \cos(2n\theta)], \quad (32)$$

for the first example was shown. A positive velocity denotes a shoreward velocity, while a negative one is a velocity directed offshore. This figure

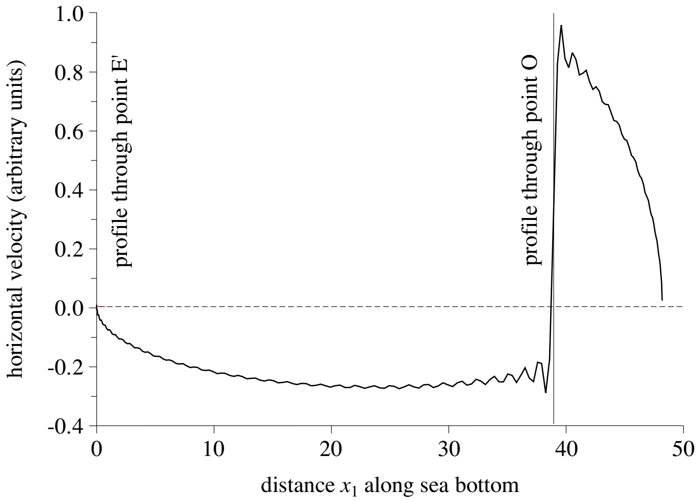

Fig. 5. Horizontal velocity in the upper bottom layer

confirms the streamline pattern given in the previous figures, namely that in the segment E′O, flow is directed offshore and in the segment OD the flow pushes water into the shore.

6. Conclusions

Sandy beaches are highly exploited but very dynamic and fragile environments. The water flow through the beach body, driven by waves, is able to transport oxygen, and hence help to maintain biological activity in the porous media. Chemical and biological reactions are fast and more abundant biota are very effective in beach self-cleaning.

This paper presents a theoretical attempt to predict the groundwater circulation induced by the set-up. The incident set-up and run-up characteristics have been determined by the methods developed in paper I. Two systems of circulations have been discovered, related to different gradients of the set-up height. For the offshore gradient, when $0 \leq x_1 \leq l_1$, the horizontal excess pressure gradient completely swamps the viscous forces in the boundary layer and carries the flow in the offshore direction. However, closer to the shore, when $l_1 \leq x_1 \leq l$, the pressure gradient is reversed and the resulting flow moves shorewards.

The proposed solution is approximate and requires experimental verification. The experimental data on the groundwater circulation are not numerous, especially on a large scale. Therefore, a new comprehensive experiment is planned and the results will be reported in a separate paper.

References

Kang H. Y., Nielsen P., 1996, *Watertable dynamics in coastal areas*, Proc. 25th Int. Coast. Eng. Conf., ASCE, 3, 4601–4612.

Li L., Barry D. A., 2000, *Wave-induced beach groundwater flow*, Adv. Water Res., 23, 325–337.

Longuet-Higgins M. S., 1983, *Wave set-up, percolation and undertow in the surf zone*, Proc. R. Soc., A390, 283–291.

Longuet-Higgins M. S., Stewart R. W., 1964, *Radiation stress in water waves: a physical discussion with applications*, Deep-Sea Res., 11, 529–562.

Massel S. R., 1976, *Gravity waves propagated over a permeable bottom*, J. Waterways, Harbors Coast. Eng. Div., ASCE, 102, WW2, 111–121.

Massel S. R., Pelinovsky E. N., 2001, *Run-up of dispersive and breaking waves on beaches*, Oceanologia, 43 (1), 61–97.

Moon P., Spencer D. E., 1961, *Field theory for engineers*, Van Nostrand Comp., Princeton, 600 pp.

Moshagen H., Torum A., 1975, *Wave induced pressures in permeable seabeds*, J. Waterways, Harbors Coast. Eng. Div., ASCE, 101, WW1, 49–57.

Saville T., 1958, *Wave runup on composite slopes*, Proc. 6th Coastal Eng. Conf., ASCE, 691–699.

Siron R., Pelletier E., Roy S., 1996, *Effects of dispersed and adsorbed oil on microalgal and bacterial communities of cold seawater*, Ecotoxicology, 5, 229–251.

Węsławski J. M., Urban-Malinga B., Kotwicki L., Opaliński K., Szymelfenig M., Dutkowski M., 2000, *Sandy coastlines – are there conflicts between recreation and natural values?*, Oceanol. Stud., 29, 5–18.