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STOKES WAVES ON SHEAR FLOW

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Abstract

Third-order Stokes waves on an arbitrary shear flow are described schematically. A comprehensive example is given for the linear velocity profile. It is shown that Stokes waves can propagate on shear flow if linear sinusoidal waves exist on this flow.

1. INTRODUCTION

The wave motion of water is intrinsically nonlinear. As the result of difficulties in the analytical treatment of nonlinear problems the nonlinear theory of waves is not well formulated. The state of the art is presented in papers [13] and [17]. Compared with the potential theory, the rotational theory of waves is poorer, as the difficulties are more serious.

However, a growing number of scientists question the assumption on the potential character of wave motion [11]. The rotational theory dates back to 1804, when Gerstner found an exact solution to nonlinear rotational waves. Dubreil-Jacotin in 1934 proved the existence of two-dimensional periodic and symmetrical waves of finite amplitude assuming small rotation of water particles in oscillatory motion. Dubreil-Jacotin's theory was extended in [6], [8], ...

Nonlinear waves on a free surface of shear flow, with its velocity varying with depth, are certainly rotational waves, and have therefore attracted the attention of theoreticians in the past twenty years. The majority of studies on the nonlinear rotational theory of waves in shear flow are concentrated, however, exclusively on long waves (solitary in [1] and [15] and cnoidal in [3], [7], ...). This is due to the fact that exact solutions for any profile of velocity can be obtained in the case of shallow water. For short waves only a few publications are known to the Author: third-order solution in [16], numerical methods for higher orders in [4], and some aspects of energy, radiation stresses and wave action in [10]. In these three papers [4], [10], [16] the linear profile of velocity has been analysed.

In this paper, third-order Stokes waves in an arbitrary shear flow are described schematically. An example is also given for the linear velocity profile. The results obtained coincide with analogous results of the classical theory of the Stokes waves and with findings of [16] for the linear velocity profile. It has been shown that third-order Stokes waves can propagate on the free surface of shear flow if linear sinusoidal waves exist on this flow.

The convergence problem for the series of stream function, free surface elevation and phase celerity is still open.

Consecultive higher-order approximations can be found by the procedure presented in this paper.

2. STOKES WAVES FOR ARBITRARY VELOCITY PROFILES

As in the classical theory of Stokes waves, we shall look for a solution to two-dimensional, periodic and symmetrical waves having constant phase velocity c. Assume the system of co-ordinates with its origin on the free surface, the x-axis in the direction of flow and the y-axis vertically upwards. In the free condition the velocity components and pressure are:

$$U=U(y),\ V=0,\ P=-\rho g y \tag{2.1}$$

The wave motion must satisfy the following equations

$$(f+u)u_x+(f'+u_y)v=-p_x/\rho \tag{2.2}$$

$$(f+u)v_x+vv_y=-p_y/\rho \tag{2.3}$$

$$u_x+v_y=0 \tag{2.4}$$

and the boundary conditions

$$(f+u)\xi'=v \quad \text{for} \quad y=\xi(x) \tag{2.5}$$

$$fp_x+(p_y-\rho g)v=0 \quad \text{for} \quad y=\xi(x) \tag{2.6}$$

$$v=0 \quad \text{for} \quad y=-H \tag{2.7}$$

in which

$$f=U(y)-c \tag{2.8}$$

u, v, p, ρ, ξ, H=velocity components, pressure, density, free surface elevation and constant depth, respectively. The apostrophe denotes ordinary differentiation with regard to the only variable, while the indices x and y denote partial derivatives.

Formula (2.5) is the common kinematic condition while eq. (2.6) describes the dynamic condition in the differential form upon the assumption of constant atmospheric pressure on the free surface.

Taking into account eq. 2.4 one can express u and v through the stream function $\Phi(x, y)$, for which

$$u=\Phi_y,\ v=-\Phi_x \tag{2.9}$$

Putting (2.9) into (2.2), (2.3), (2.5), (2.6) and (2.7) and reducing p yields

$$(f+\Phi_y)\Delta\Phi_x-\Phi_x(f''+\Delta\Phi_y)=0 \tag{2.10}$$

$$(f+\Phi_y)\xi'=-\Phi_x \quad \text{for} \quad y=\xi(x) \tag{2.11}$$

$$(f+\Phi_y)[(f+\Phi_y)\Phi_{xy}-\Phi_x(f'+\Phi_{yy})]-\Phi_x[g+\Phi_x\Phi_{xy}-(f+\Phi_y)\Phi_{xx}]=0 \quad (2.12)$$

for $y=\xi(x)$, $\Phi_x=0$ for $y=-H$ (2.13)

in which $\Delta\Phi=\Phi_{xx}+\Phi_{yy}$

It is worthwhile to mention that similar to the Dubreil-Jacotin theory one can obtain the simpler form

$$\Delta\Phi(x,y)=\Omega(\Phi)$$

in which $\Omega(\Phi)=U'(y)$ if the flow is free. The function denotes the stream function for the resultant motion due to waves and flow. The boundary conditions become

$$\Phi_y\xi'=-\Phi_x \quad \text{for} \quad y=\xi(x)$$

$$(\Phi_x^2+\Phi_y^2)+2g\xi=f^2-gH \quad \text{for} \quad y=\xi(x)$$

$$\Phi_x=0 \quad \text{for} \quad y=-H$$

Our analysis, however, will be based on the relationships (2.10) ... (2.13), in which the contribution of the velocity distribution $U(y)$ is more apparent.

Using the Stokes primary method (while the secondary method employs conformal mapping) we will expand the unknown functions Φ, ξ and phase celerity c in power series in respect of the small parameter ε

$$\Phi(x,y)=\varepsilon\Phi_1(x,y)+\varepsilon^2\Phi_2(x,y)+\varepsilon^3\Phi_3(x,y)+\dots$$

$$\xi(x)=\xi_0(x)+\varepsilon\xi_1(x)+\varepsilon^2\xi_2(x)+\varepsilon^3\xi_3(x)+\dots \quad (2.14)$$

$$c=c_0+\varepsilon c_1+\varepsilon^2 c_2+\varepsilon^3 c_3+\dots$$

Upon consideration of the following free-surface relationship

$$F(x,y)=F(x,0)+\xi F_y(x,0)+\tfrac{1}{2}\xi^2F_{yy}(x,0)+\tfrac{1}{6}\xi^3F_{yyy}(x,0)+\dots \quad (2.15)$$

in which ξ is given by (2.14) one can formulate the boundary problems for Φ_n, ξ_n and c_n.

The introduction of (2.14), (2.15) to (2.10) ... (2.13) gives:

a) For the zeroth order

$$f_{00}\xi_0'=0 \text{ in which } f_{00}=f_0(0)=U(0)-c_0 \quad (2.16)$$

(from (2.16) one can take $\xi_0\equiv 0$, with the x-axis on the quiescent horizontal free surface)

b) For the first order

$$f_0\Delta\Phi_{1x}-U''\Phi_{1x}=0 \quad \text{for} \quad -H\leqslant y\leqslant 0,\ -\infty<x<\infty \quad (2.17)$$

$$f_0\xi_1'+\Phi_{1x}=0 \quad \text{for} \quad y=0 \quad (2.18)$$

$$f_0(U'\Phi_{1x}-f_0\Phi_{1xy})+g\Phi_{1x}=0 \quad \text{for} \quad y=0 \quad (2.19)$$

$$\Phi_{1x}=0 \quad \text{for} \quad y=-H \quad (2.20)$$

in which

$$f_0 = U(y) - c_0 \tag{2.21}$$

In the search for solutions to (2.10) ... (2.13) as symmetrical and periodic waves we will present (2.17) ... (2.20) as sinusoidal waves, i.e.

$$\Phi_1 = \varphi_1(y) \cos kx \tag{2.22}$$

Accordingly, relationships (2.17) ... (2.20) will take the form

$$f_0 \varphi_1'' - [U'' + k^2 f_0] \varphi_1 = 0 \quad \text{for} \quad -H \leqslant y \leqslant 0 \tag{2.23}$$

$$f_0 \xi_1' = k \varphi_1 \sin kx \quad \text{for} \quad y = 0 \tag{2.24}$$

$$f_0 [U' \varphi_1 - f_0 \varphi_1'] = -g \varphi_1 \quad \text{for} \quad y = 0 \tag{2.25}$$

$$\varphi_1 = 0 \quad \text{for} \quad y = -H \tag{2.26}$$

Eq. (2.23) and the boundary conditions (2.24) ... (2.26) can also be obtained through linearization of (2.10) ... (2.13), as in the linear theory of waves [19].

Even though simple, eq. (2.23) can be solved explicity for only a very narrow class of the functions $U(y)$ which satisfy the condition

$$k^2 + U''/f_0 = my^\beta$$

in which m is an arbitrary number, while β assumes the values 0,2, $4n/(1-2n)$ for $n = \pm 1, \pm 2, \pm 3, \ldots$ In all remaining cases eq. (2.23) can be solved only in approximation. A certain simple technique basing on the approximation of an arbitrary velocity profile by a broken line, with an optimum condition, is discussed in [19].

Considering the system (2.23) ... (2.26) as an eigenvalue problem for any $U(y)$, one encounters some peculiarities.

Proof the existence of the solution (2.22) with the eigen function $\varphi_1(y)$ and the eigenvalue C_0 satisfying (2.23) ... (2.26) is difficult, but can be found for a special class of velocity profiles. It is shown in [20] that the solution (2.22) exists on the free surface of a shear flow, the velocity of which increases with y, but shows down the closer it is to the free surface, i.e. $U'(y) > 0$, $U''(y) < 0$. Then a wave with any wave number k can propagate in the direction of flow, while waves with wave numbers less than a certain critical value (i.e. sufficiently long waves) can propagate in the opposite direction. Apart from this, the phase velocity of any wave is real and bounded.

Assume that the velocity profile permits the existence of the solution (2.22) in the first approximation, which is never equal to zero in the interval $-H \leqslant y \leqslant 0$. In other words, assume that the following functions are known

$$\varphi_1 = A_{11} \varphi_{11}(y, k) + B_{11} \varphi_{12}(y, k) \tag{2.27}$$

in which φ_{11}, φ_{12} are two independent solutions of the eq. (2.23) satisfying the condition (2.26) and

$$c_0 = c_0(U, k) \tag{2.28}$$

From the condition (2.24) one has

$$\xi_1=(\varphi_1)_{y=0}\cos kx/[c_0-U(0)]=a_1\cos kx \tag{2.29}$$

with zero as the constant of integration, so that the x-axis lies on the undisturbed free surface.

c) For the second order

It can be checked that in the second approximation eq. (2.10) becomes

$$f_0\Delta\Phi_{2x}-U''\Phi_{2x}=\Phi_{1x}\Delta\Phi_{1y}-\Phi_{1y}\Delta\Phi_{1x} \tag{2.30}$$

The conditions (2.11) ... (2.13) will take the form

$$f_0\xi_2'+\Phi_{2x}=-\xi_1\Phi_{1xy}-(U'\xi_1+\Phi_{1y})\xi_1' \quad \text{for} \quad y=0 \tag{2.31}$$

$$f_0(U'\Phi_{2x}-f_0\Phi_{2xy})+g\Phi_{2x}=f_0[2\Phi_{1x}\Phi_{1xy}+U'\xi_1\Phi_{1xy}+\Phi_{1x}\Phi_{1xx}+$$
$$-(U''\xi_1+\Phi_{1yy})\Phi_{1x}+f_0\xi_1\Phi_{1xyy}]-U'\Phi_{1x}(U'\xi_1+\Phi_{1y})-g\xi_1\Phi_{1xy} \tag{2.32}$$

for $y=0$, $\Phi_{2x}=0$ for $y=-H$ (2.33)

Taking into account (2.22) ... (2.25), after a number of simplifications one obtains (2.30) ... (2.32) in the form

$$f_0\Delta\Phi_{2x}-U''\Phi_{2x}=\frac{k}{2}(\varphi_1'\Omega_1-\varphi\Omega_1')\sin 2kx \tag{2.34}$$

$$f_0\xi_2'+\Phi_{2x}=\frac{k\varphi_1}{2f_0^2}(\varphi_1 U'-2f_0\varphi_1')\sin 2kx \quad \text{for} \quad y=0 \tag{2.35}$$

$$f_0(U'\Phi_{2x}-f_0\Phi_{2xy})+g\Phi_{2x}=\frac{k\varphi_1}{2f_0}[f_0^2\varphi_1''-3g\varphi_1'+(2k^2f_0^2-U'^2)\varphi_1]\sin 2kx$$
$$\text{for } y=0 \tag{2.36}$$

in which

$$\Omega_1=\varphi_1 U''/f_0$$

The appearance of $\sin 2kx$ on the right hand sides of the relationships (2.34) ... (2.36) indicates that the problem (2.33) ... (2.36) can be solved in the form

$$\Phi_2=\varphi_2(y)\cos 2kx, \quad c_1=0 \tag{2.37}$$

Eq. (2.34) takes the form

$$f_0\varphi_2''-[U''+(2k)^2f_0]\varphi_2=R_2 \tag{2.38}$$

in which

$$R_2=\frac{1}{4}(\varphi_1\Omega_1'-\varphi_1'\Omega_1)=\frac{1}{4}\varphi_1^2\left(\frac{U''}{f_0}\right)' \tag{2.39}$$

It can be checked that eq. (2.38) has the following general solution satisfying the relationship (2.33)

$$\varphi_2=A_{21}\varphi_{21}+B_{21}\varphi_{22}+\varphi_{22}\int_{-H}^{y}\frac{\varphi_{21}R_2}{W_2}dy-\varphi_{21}\int_{-H}^{y}\frac{\varphi_{22}R_2}{W_2}dy \tag{2.40}$$

in which

$$W_2=\varphi_{21}\varphi'_{22}-\varphi'_{21}\varphi_{22} \tag{2.41}$$

and φ_{21}, φ_{22} are two independent solutions of the homogeneous equations related to eq. (2.38), which is identical with eq. (2.30) if k is replaced by $2k$. Thus, one has

$$\varphi_{21}=\varphi_{11}(y,2k),\quad \varphi_{22}=\varphi_{12}(y,2k) \tag{2.42}$$

Substituting eqs. (2.37), (2.40) in eq. (2.35) and integrating the latter with regard to x one obtains

$$\xi_2=a_2\cos 2kx \tag{2.43}$$

in which

$$a_2=-\left[\frac{\varphi_1}{4f_0^2}(\varphi_1 U'-2\varphi'_1 f_0)+\varphi_2\right]/f_0 \quad \text{for} \quad y=0 \tag{2.44}$$

The condition (2.36) is required to determine the constants A_{21}, B_{21}, A_{11} and B_{11}.

d) For the third order

Taking into account eqs. (2.22) and (2.37) one obtains the following third-order relationship from eq. (2.10)

$$f_0\Delta\Phi_{3x}-U''\Phi_{3x}=-kR_{31}\sin kx-3kR_{33}\sin 3kx \tag{2.45}$$

where

$$R_{31}=\varphi_2 Q'_1+\tfrac{1}{2}\varphi'_2 Q_1-\varphi'_1 Q_2-\tfrac{1}{2}\varphi_1 Q'_2+c_2 Q_1, \tag{2.46}$$
$$R_{33}=\tfrac{1}{3}(\varphi_2 Q'_1-\tfrac{1}{2}\varphi'_2 Q_1-\varphi'_1 Q_2+\tfrac{1}{2}\varphi_1 Q'_2),\quad Q_n=\varphi''_n-k^2\varphi_n;\qquad n=1,2.$$

The general solution of eq. (2.45) satisfying the condition at the bed can be presented in the form:

$$\Phi_3=\varphi_{31}(y)\cos kx+\varphi_{33}(y)\cos 3kx \tag{2.47}$$

in which φ_{31}, φ_{33} must satisfy the following relationships

$$f_0\varphi''_{31}-[U''+k^2 f_0]\varphi_{31}=R_{31} \tag{2.48}$$

$$f_0\varphi''_{33}-[U''+(3k)^2 f_0]\varphi_{33}=R_{33} \tag{2.49}$$

It thus results that

$$\varphi_{3n}=A_{31}^{(n)}\varphi_{31}^{(n)}+B_{32}^{(n)}\varphi_{32}^{(n)}+\varphi_{32}^{(n)}\int_{-H}^{y}\frac{\varphi_{31}^{(n)}R_{3n}}{W_{3n}}dy-\varphi_{31}^{(n)}\int_{-H}^{y}\frac{\varphi_{32}^{(n)}R_{3n}}{W_{3n}}dy \tag{2.50}$$

in which

$$\varphi_{31}^{(n)}=\varphi_{11}(y,nk),\quad \varphi_{32}^{(n)}=\varphi_{12}(y,nk),\quad W_{3n}=\varphi_{31}^{(n)}\varphi'^{(n)}_{32}-\varphi'^{(n)}_{31}\varphi_{32}^{(n)},\quad n=1,3 \tag{2.51}$$

Condition (2.11) reads: for $y=0$

$$f_0\xi_3'+\Phi_{3x}=-\{(U'\xi_1+\Phi^{1y})\xi^{2\prime}+)U'\xi^2+\tfrac{1}{2}\xi_1^2U''+\Phi_{2y}+\xi_1\Phi_{1yy}-c_2)\xi_1'+$$
$$+\xi_1\Phi_{2xy}+\xi_2\Phi_{1xy}+\tfrac{1}{2}\xi_1^2\Phi_{1xyy}\}$$

As a consequence of (2.22), (2.27), (2.37), (2.40), (2.43), (2.47) and (2.50) one obtains

$$f_0\xi_3'=k\,[\varphi_{31}+U'a_1a_2+\tfrac{1}{2}a_1(2\varphi_2'+2M_2-M_1)+\tfrac{1}{8}a_1^2\varphi_1'']\sin kx+$$
$$+k\,[3\varphi_{33}+a_2(U'a_1+2\varphi_1')+\tfrac{1}{2}a_1(M_1+2\varphi_2')+\tfrac{1}{8}a_1^2\varphi_1'']\sin 3kx \tag{2.52}$$

in which

$$M_1=(U'a_2+\varphi_2'+\tfrac{1}{2}a_1\varphi_1''+\tfrac{1}{4}U''a_1^2)_{y=0}$$
$$M_2=(\tfrac{1}{4}U''a_1^2+\tfrac{1}{2}a_1\varphi_1''-c_2)_{y=0} \tag{2.53}$$

It is easily seen that ξ_3 should be

$$\xi_3=a_{31}\cos kx+a_{33}\cos 3kx \tag{2.54}$$

in which

$$a_{31}=-[\varphi_{31}+U'a_1a_2+\tfrac{1}{2}a_1(2\varphi_2'+2M_2-M_1)+\tfrac{1}{8}a_1^2\varphi_1'']_{y=0}$$
$$a_{33}=-\left[\varphi_{33}+\frac{1}{3}a_2(U'a_1+2\varphi_1')+\frac{1}{6}a_1(M_1+2\varphi_2')+\frac{a_1^2}{24}\varphi_1''\right]_{y=0} \tag{2.55}$$

Note that condition (2.12) now reads

$$f_0(U'\Phi_{3x}-f_0\Phi_{3xy})+g\Phi_{3x}=f_0(A_3-B_3)+(U'\xi_1+\Phi_{1y})(A_2-B_2)+f_0X_2\Phi_{1xx}+$$
$$+(f_0\Phi_{1xy}-U'\Phi_{1x})(f_2+Y_2)-gX_3+\Phi_{1x}(D_2-E_2) \tag{2.56}$$

in which

$$X_2=\Phi_{2x}+\xi_1\Phi_{1xy},\quad X_3=\xi_1\Phi_{2xy}+\xi_2\Phi_{1xy}+\tfrac{1}{2}\xi_1^2\Phi_{1xyy},$$
$$Y_2=\Phi_{2y}+\xi_1\Phi_{1yy},\quad f_2=U'\xi_2+\tfrac{1}{2}U''\xi_1^2-c_2,\quad A_2=f_0X_{2y}+(U'\xi_1+\Phi_{1y})\Phi_{1xy},$$
$$A_3=f_0X_{3y}+(U'\xi_1+\Phi_{1y})X_{2y}+(f_2+Y_2)\Phi_{1xy},\quad B_2=U'X_2+(U''\xi_1+\Phi_{1yy})\Phi_{1x},$$
$$B_3=U'X_3+(U''\xi_1+\Phi_{1yy}')X_2+(f_{2y}+Y_{2y})\Phi_{1x},\quad D_2=f_0X_{2x}+(U'\xi_1+\Phi_{1y})\Phi_{1xx},$$
$$E_2=\Phi_{1x}\Phi_{1xy} \tag{2.57}$$

Summarizing the relationships obtained for C_0 one can present the right hand side of eq. (2.56) in the form

$$S_1\sin kx+S_3\sin 3kx \tag{2.58}$$

$$S_1=k\,\{f_0(U'T_1-f_0T_1'+T_2'T_3-T_3'T_2+\varphi_1T_4'-\varphi_1'T_4+\varphi_1'c_2)+$$
$$+T_2[U'T_3-f_0T_3'+4(T_2'\varphi_1-T_2\varphi_1')]+T_7(T_4-c_2)+gT_1+$$
$$\left.+\frac{k^2}{2}\varphi_1\left(2T_2\varphi_1+\frac{3}{2}\varphi_1\varphi_1'-\frac{1}{2}f_0T_3\right)\right\}_{y=0} \tag{2.59}$$

$$S_3=k\{f_0[U'T_6-f_0T_6'+T_2'T_3-T_2T_3'+\varphi_1T_5'-\varphi_1'T_5]+T_5T_7+gT_6+$$
$$+T_2[U'T_3-f_0T_3'+4(T_2'\varphi_1-T_2\varphi_1')]+$$
$$+\frac{k^2}{2}\varphi_1\left(2T_2\varphi_1-\frac{1}{2}\varphi_1\varphi_1'+\frac{3}{2}f_0T_3\right)\Bigg\}_{y=0} \qquad (2.60)$$

$$T_1=(a_1\varphi_2'-\tfrac{1}{2}a_2\varphi_1'+\tfrac{1}{8}a_1^2\varphi_1'')_{y=0}, \quad T_6=(T_1+a_1\varphi_1')_{y=0},$$
$$T_2=\tfrac{1}{4}(U'a_1+\varphi_1')_{y=0}, \quad T_3=(4\varphi_2+a_1\varphi_1')_{y=0}$$
$$T_4=(T_5-\varphi_2'-U'a_2)_{y=0}, \quad T_7=(U'\varphi_1-f_0\varphi_1')_{y=0}, \qquad (2.61)$$
$$T_5=(\tfrac{1}{8}U''a_1^2+\tfrac{1}{4}a_1\varphi_1''+\tfrac{1}{2}\varphi_2'+\tfrac{1}{2}U'a_2)_{y=0}$$

It can be seen that the term with sin kx is simultaneously included in eq. (2.45) and conditions (2.52), (2.56). Contrary to (2.45) and (2.52) in which this term does not bring about any serious obstacle in the search for solutions, eq. (2.56) embodies S_1 sin kx of another, resonant character, often encountered in the nonlinear theory mechanical oscillations. In our case the solution of eq. (2.56) cannot be solved as S_1 is not zero.

The Stokes method (known as Poincaré's method in the theoretical-mechanics) enables this obstacle to be overcome. The basic idea of this consists in the expansion in power series of the small parameter ε, in respect of which other unknown function are expanded. In fact, C_2 can be chosen so that S_1 is zero. In this case C_2 will read

$$C_2=\{[f_0(U'T_1-f_0T_1'+T_2'T_3-T_2T_3'+\varphi_1T_4'-\varphi_1'T_4)+T_4T_7+gT_1+$$
$$+T_2(U'T_3-f_0T_3'+4\varphi_1T_2'-4\varphi_1'T_2)+$$
$$+\tfrac{1}{2}k^2\varphi_1(2T_2\varphi_1+\tfrac{3}{2}\varphi_1\varphi_1'-f_0T_3)]/(T_7-f_0\varphi_1')\}_{y=0} \qquad (2.62)$$

In continuation of the above procedure one can obtain higher order approximations for the Stokes wave in shear flows. However, the third order should be sufficient, as the higher approximations have limited application (see § 5.3.2 in [5]).

The ultimate formulae for Φ, ξ, and c in the third order approximation read

$$\Phi=[\varepsilon\varphi_1(y)+\varepsilon^3\varphi_{31}(y)]\cos kx+\varepsilon^2\varphi_2(y)\cos 2kx+\varepsilon^3\varphi_{33}(y)\cos 3kx$$
$$\xi=(\varepsilon a_1+\varepsilon^3a_{31})\cos kx+\varepsilon^2a_2\cos 2kx+\varepsilon^3a_{33}\cos 3kx$$
$$c=c_0+\varepsilon^2c_2 \qquad (2.63)$$

in which φ_i, φ_{3n}, a_i, a_{3n}, c_0, c_2 are given by (2.27), (2.40), (2.50), (2.29), (2.44), (2.55), (2.28) and (2.62), respectively.

Denoting $\varepsilon a_1+\varepsilon^3a_{31}$ by the amplitude a it is possible to present explicit relationships for the coefficients in eq. (2.63), as functions of a and $U(y)$.

Thus, we have proved that the Stokes third-order waves can propagate on the free surface of a shear flow – provided the motion consists of sinusoidal waves of infinitely small amplitude that satisfy the linearized equation and its boundary conditions (2.23) ... (2.26).

The proof of the convergence of the series (2.14) is even more difficult than in the classical theory by Stokes. However, one can hope that this proof can be derived for a certain class of the functions $U(y)$ if the parameter ε is small enough, ε being wave amplitude or steepness. It can be shown that the Stokes waves in a shear flow are unstable with regard to small perturbation as they contain the classical Stokes waves, the instability of which has been proved by Benjamin in his outstanding work [2] and confirmed in other experimental, as well as theoretical studies [9], [18], ...

It is worthwhile to mention that the phase velocity in Dubreil-Jacotin's theory can by computed by Airy's formula

$$c=\left[\frac{g}{k}\operatorname{th}kH\right]^{1/2} \tag{2.64}$$

with accuracy to the first-order amplitude. In our case, as for the classical Stokes waves, the phase velocity.

C_0 is determined with an accuracy of squared amplitude. This discrepancy does not stem from small vorticity of the resultant motion of water particles in Dubreil--Jacotin's theory, but rather from the method used in its proof (No restrictions have been put on vorticity i.e. on the velocity gradient $U'(y)$ in our case).

The nonlinearity of the effect of flow on the Stokes waves is visible not only in the formulae for velocity C_0, C_2 and amplitude a, a_2, a_3, a_3 but also through the dependence of wave motion on the direction of wave propagation. This effect cannot be determined clearly for any velocity profile. This is discussed later by expansion of the above procedure for the linear velocity profile, which can be treated as a first approximation for an arbitrary velocity profile.

3. STOKES WAVES IN FLOW WITH LINEAR VELOCITY PROFILE

Without loss of generality, the problem can be discussed for the following linear velocity profiles

$$U=G(y+H) \tag{3.1}$$

a) Zeroth order

One has $\xi_0 \equiv 0$, as for any velocity profile.

b) First order

The solution satisfying the solution (2.23) and (2.26) is

$$\Phi_1=A_1\operatorname{sh}k(y+H)\cos kx \tag{3.2}$$

Substitution of (3.2) into (2.25) gives

$$\mathscr{C}_0^2=\frac{1}{k}(g+\mathscr{C}_0\ \operatorname{th}kH, \quad \text{where } \mathscr{C}_0=GH-c_0 \tag{3.3}$$

The relation (2.29) can be written

$$\xi_1 = a_1 \cos kx, \quad a_1 = -\frac{A_1}{\mathscr{C}_0} \operatorname{sh} kH \tag{3.4}$$

c) Second order

Due to (3.1), (2.39) yields $R_2 = 0$. The solution (2.40) satisfying (2.38) and (2.13) becomes

$$\Phi_2 = A_2 \operatorname{sh} 2k(y+H) \cos 2kx, \quad c_1 = 0 \tag{3.5}$$

After a number of simplifications the condition (2.36) gives

$$A_2 = \frac{-kA_1^2}{8\mathscr{C}_0 \operatorname{sh}^2 kH}\left[3 - \frac{G(3g+2G\mathscr{C}_0)}{k^2\mathscr{C}_0^3} \operatorname{sh}^2 kH\right] \tag{3.6}$$

A combination of (3.2), (3.5) and (2.44) yields

$$\xi_2 = \frac{ka_1^2}{4}\left[2\operatorname{cth} kH + \frac{3\operatorname{ch} kH}{\operatorname{sh}^3 kH} - \frac{G}{k\mathscr{C}_0}\left(1 + \frac{3g+2G\mathscr{C}_0}{k\mathscr{C}_0^2} \operatorname{cth} kH\right)\right] \cos 2kx \tag{3.7}$$

d) Third order

Basing on (3.2) and (3.5) we obtain $Q_1 = 0$ and $Q_2 = 0$. Thus the right hand side of eq. (2.45) is zero. The solution that satisfies eq. (2.45) and the condition of the bed reads

$$\Phi_3 = A_3 \operatorname{sh} 3k(y+H) \cos 3kx \tag{3.8}$$

The ultimate formulae for a_3, a_{33}, s_1 and s_3 can result from (2.55), (2.59), (2.60) and (2.61). Due to the sophisticated form of these formulae it is only C_2 which will be written down as the consequence of (2.62)

$$c_2 = -\frac{[2g(8+\operatorname{ch} 4kH) + G\mathscr{C}_0(12+\operatorname{sh}^2 kH)]}{16(2g+G\mathscr{C}_0)\operatorname{sh}^4 kH} k^2 a_1^2 \mathscr{C}_0 + \frac{ka_1^2 G(g+G\mathscr{C}_0)}{8(2g+G\mathscr{C}_0)} \Theta \tag{3.9}$$

in which

$$\Theta = 2\frac{1+\operatorname{ch} 2kH(1+\operatorname{ch}^2 kH)}{\operatorname{sh}^2 kH \operatorname{sh} 2kH} + \frac{g}{g+G\mathscr{C}_0}\left[\frac{\operatorname{ch} kH(2+\operatorname{ch}2kH)}{\operatorname{sh}^3 kH} - \frac{G}{k\mathscr{C}_0}\right] \tag{3.10}$$

$$-\frac{G}{k\mathscr{C}_0} + \frac{3g+2G\mathscr{C}_0}{k\mathscr{C}_0^2}\left[\frac{g}{g+G\mathscr{C}_0}\left(\frac{\operatorname{ch} 2kH}{\operatorname{sh}^2 kH} - \frac{G\operatorname{ch} kH}{k\mathscr{C}_0 \operatorname{sh} kH}\right) + \frac{2}{\operatorname{sh}^2 kH} - \frac{G\operatorname{ch} kH}{k\mathscr{C}_0 \operatorname{sh} kH}\right]$$

In the summary one obtains:

In the second order

$$\Phi = \varepsilon\Phi_1 + \varepsilon^2\Phi_2 = \varepsilon A_1 \operatorname{sh} k(y+H)\cos kx + \varepsilon^2 A_2 \operatorname{sh} 2k(y+H)\cos 2kx$$

$$\xi = \varepsilon a_1 \cos kx + \varepsilon^2 a_2 \cos 2kx$$

Denoting εa by a one obtains

$$\Phi=-\frac{a\mathscr{C}_0}{\operatorname{sh}kH}\operatorname{sh}k(y+H)\cos kx-\frac{a^2\mathscr{C}_0k}{8\operatorname{sh}^4kH}\left[3-\frac{G(3g+G\mathscr{C}_0)}{k^2\mathscr{C}_0^3}\operatorname{sh}^2kH\right]\times$$

$$\times\operatorname{sh}2k(y+H)\cos 2kx \qquad (3.11)$$

$$\xi=a\cos kx+\frac{a^2k}{4}\operatorname{cth}kH\left[2+\frac{3}{\operatorname{sh}^2kH}-\frac{G}{k\mathscr{C}_0}\left(\operatorname{th}kH+\frac{3g+2G\mathscr{C}_0}{k\mathscr{C}_0^2}\right)\right]\cos 2kx \qquad (3.12)$$

In the third-order approximation

$$c=c_0-\frac{2g(8+\operatorname{ch}4kH)+G\mathscr{C}_0(12+\operatorname{sh}^4kH)}{16(2g+G\mathscr{C}_0)\operatorname{sh}^4kH}a^2k^2\mathscr{C}_0+\frac{a^2kG(g+G\mathscr{C}_0)\,\Theta}{8(2g+G\mathscr{C}_0)} \qquad (3.13)$$

in which C_0 is given by eq. (3.3).

The dispersion relationship, with C^2 on the left-hand side, can often be useful. Taking into account

$$c^2=(-\mathscr{C}_0+GH+\varepsilon^2c_2)^2\approx\mathscr{C}_0^2\left(1-\frac{2GH}{\mathscr{C}_0}-\frac{2\varepsilon^2c_2}{\mathscr{C}_0}\right)$$

one can write

$$c^2=\mathscr{C}_0^2\left\{1-\frac{2GH}{\mathscr{C}_0}+\frac{a^2k^2[2g(8+\operatorname{ch}4kH)+G\mathscr{C}_0(12+\operatorname{sh}^2kH)]}{8(2g+G\mathscr{C}_0)\operatorname{sh}^4kH}-\right.$$

$$\left.-\frac{a^2kG(g+G\mathscr{C}_0)}{4(2g+G\mathscr{C}_0)}\Theta\right\} \qquad (3.14)$$

In the case of $G=0$ the formulae for ξ and C^2 read

$$\xi=a\cos kx+\frac{a^2k}{2}\operatorname{cth}kH\left(1+\frac{3}{2\operatorname{sh}^2kH}\right) \qquad (3.15)$$

$$c^2=c_{00}^2\left[1+\frac{a^2k^2(8+\operatorname{ch}4kH)}{8\operatorname{sh}^4kH}\right],\quad c_{00}^2=\frac{g}{k}\operatorname{th}kH \qquad (3.16)$$

They are classical formulae for the Stokes waves [17]. Formulae (3.11), (3.12) and (3.13) coincide with the results given in [16] and derived in other ways. However, the former, are much simpler. This particularly is true for the dispersion relation (3.13). They indicate explicitly the contribution of flow parameters to the stream function, amplitude and phase velocity. This is analysed in [16].

From eq. 3.31 one obtains

$$\mathscr{C}_0=[G\operatorname{th}kH\pm(G^2\operatorname{th}^2kH+4gk\operatorname{th}kH)^{1/2}]/2k$$

$$\text{or } c_0-U(0)=[-G\operatorname{th}kH\pm(G^2\operatorname{th}kH+4gk\operatorname{th}kH)^{1/2}]/2k \qquad (3.17)$$

Basing on eqs. (3.4) and (3.17) it can be seen that the waves propagating with the flow (i.e. $C_0-U(0)>0$) have a velocity of $|C_0-U(0)|<|C_{00}|$, so that the amplitude increases. For the opposite direction of wave propagation this is $|C_0-U(0)|>|C_{00}|$, hence the amplitude decreases.

This explains the commonly-known fact that the flow intensifies the accompanying (downstream) waves and dampens the opposite (upstream) waves.

A more detailed analysis also indicates that the downstream waves are steeper and longer than the respective waves in the Stokes theory. For the upstream waves the picture is the opposite. In addition, it can be shown that the steppness of the downstream waves increases with the flow gradient. It is interesting that the term $\cos 2kx$ can be omitted in eq. (3.12) for the free surface elevation, notably for certain (although substantial) gradients G. In these cases sinusoidal waves can describe the free surface oscillations with an accuracy of the third-order of amplitude.

Finally, another peculiarity of the linear velocity profile. Formulae (3.2), (3.5) and (3.8) show that the three first orders of approximation for the stream function satisfy Laplace's equation, so that the wave motion is potential by the third-order approximation inclusively.

We will prove that for the flow profile (3.1) the wave motion is potential in general. In fact, the two-dimensional resultant motion of an ideal, incompressible and barotropic liquid due to potential external forces, satisfies the Helmholtz equation [12]

$$d\Omega/dt=0$$

in which Ω is a vector of vorticity.

Assume that this resultant motion is generated in a shear flow, characterized by parameters (2.1) in undisturbed condition. Denoting by $\vec{\Omega}_1$, the vorticity in the wave motion, one can write

$$\Omega=U'+\Omega_1 \quad \text{hence} \quad \frac{d\Omega_1}{dt}+vU''=0 \tag{3.18}$$

in which v is the velocity component in the wave motion.

From eq. (3.18) it follows that the wave motion will be potential only if the velocity profile is linear, as in this case one has $\Omega_1=\text{const}=0$. Accordingly, for all consecutive approximations one obtains

$$\Phi_n=A_n \operatorname{sh} nk(y+H)\cos nkx \qquad n=1,2,3,\ldots$$

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