Papers

On the resistance law in a shallow sea

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Bottom stress
Shallow sea

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Abstract

Assuming a two-layer structure of turbulence in a homogeneous shallow sea, expressions describing the geostrophic coefficient of bottom friction u_*/G and the angle between the vector of geostrophic current and the vector of tangential stress at the bottom were obtained as functions of non-dimensional parameters.

The generally accepted method of determining the effect of bottom friction in a shallow sea is based on the use of the square law of friction, in which the coefficient of bottom friction is a constant quantity or is defined by flow parameters [4]. Besides, it is known that in a shallow sea with neutral stratification, the vertical distribution of velocity and the turbulence characteristics determining the magnitude and direction of bottom stress are functions of normalized with respect to sea water density components of wind tangential stress at the free surface (τ_x^a/ρ, τ_y^a/ρ), velocity of the geostrophic current (G), depth (H), Coriolis parameter (f), and the sea bottom roughness parameter (z_0). According to the π-theorem of the similarity theory, four non-dimensional combinations may be formed out of the six above-mentioned parameters: $\tau_x^a/\rho G^2$, $\tau_y^a/\rho G^2$, G/fz_0, H/z_0. It may be expected that all turbulence characteristics in a shallow sea with neutral stratification, including the magnitude and direction of the bottom stress vector, will be related to the external defining parameters only by the above-metioned non-dimensional combinations.

The aim of this paper is to obtain expressions describing the geostrophic coefficient of bottom friction u_*/G and the angle α between the vector of geostrophic current and the direction of tangential stress at the bottom (veering angle) as a function of $\tau_x^a/\rho G^2$, $\tau_y^a/\rho G^2$, G/fz_0, H/z_0.

We take the traditional description of velocity and structure of turbulence in a shallow homogeneous sea as our starting point. We thus assume that in the thin layer close to the bottom the vertical distribution of velocity is determined by the logarithmic law (correspondingly, the vertical eddy-viscosity coefficient is a linear function of the vertical coordinate), and at the interface between the near-bottom logarithmic layer and the layer above it, the coefficient of turbulent viscosity and the momentum flux are continuous. In this case, the equation of motion and the boundary conditions describing the vertical velocity distribution above the logarithmic layer take the following form:

$$\left.\begin{aligned} v-G\sin\alpha+\frac{1}{f}\frac{\partial}{\partial z}k\frac{\partial u}{\partial z}=0, \\ u-G\cos\alpha-\frac{1}{f}\frac{\partial}{\partial z}k\frac{\partial v}{\partial z}=0, \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} k\frac{\partial u}{\partial z}=\tau_x^a/\rho\,; \quad k\frac{\partial v}{\partial z}=\tau_y^a/\rho \quad \text{at } z=H, \\ k\frac{\partial u}{\partial z}=\tau_x^0/\rho\,; \quad k\frac{\partial v}{\partial z}=\tau_y^0/\rho \quad \text{at } z=h, \end{aligned}\right\} \tag{2}$$

where:

u, v – current velocity components in the direction of x and y axes respectively (x axis is directed along the vector of bottom friction),

G – geostrophic current velocity modulus $\left(G=\frac{1}{\rho f}\sqrt{\left(\frac{\partial p}{\partial x}\right)^2+\left(\frac{\partial p}{\partial y}\right)^2}\right.$, where, ρ – water density, f – Coriolis parameter, and p – hydrostatic pressure),

k – eddy-viscosity coefficient,

h – thickness of the logarithmic bottom layer,

z – axis directed vertically upwards.

The origin of the coordinate system is situated at the sea bottom.

Let us assume that the eddy-viscosity coefficient k in the layer $h \leqslant z \leqslant H$ is constant and, according to the continuity condition at a depth $z=h$, equals:

$$k=\kappa u_* h\,, \tag{3}$$

where: u_* = frictional velocity, κ = von Kármán constant.

We now use the above-mentioned logarithmic velocity profile in the near-bottom layer and the continuity condition of velocity at a level $z=h$. As a result we have:

$$\left.\begin{aligned} \frac{u_*}{\kappa}\ln\frac{h}{z_0}-G\cos\alpha=\frac{u_*}{\kappa}F_u(h) \\ -G\sin\alpha=\frac{u_*}{\kappa}F_v(h) \end{aligned}\right\} \tag{4}$$

where:

$$\frac{u_*}{\kappa}F_u(h)=\frac{1}{f}\left.\frac{\partial\tau_y/\rho}{\partial z}\right|_{z=h}; \quad \frac{u_*}{\kappa}F_v(h)=-\frac{1}{f}\left.\frac{\partial\tau_x/\rho}{\partial z}\right|_{z=h}. \tag{5}$$

We determine the thickness h of the logarithmic layer by assuming that is proportional to u_*/f, *ie*:

$$h=c_1\frac{u_*}{f}; \quad c_1=\text{numerical constant}. \tag{6}$$

Comparing expressions (4) and (5), and taking into account (6), after certain transformations, we obtain:

$$\left.\begin{aligned}&\ln R_0=B-\ln(u_*/G)+\sqrt{\frac{\kappa^2}{(u_*/G)^2}-A^2},\\ &\sin\alpha=-\frac{A}{\kappa}\frac{u_*}{G},\end{aligned}\right\} \tag{7}$$

where:

R_0 – Rossby number ($R_0=G/f_{z_0}$),

$$\left.\begin{aligned}&A=F_v(h)=-\frac{\kappa}{fu_*}\frac{\partial\tau_x/\rho}{\partial z}\bigg|_{z=h},\\ &B=F_u(h)-\ln c_1=\frac{\kappa}{fu_*}\frac{\partial\tau_y/\rho}{\partial z}\bigg|_{z=h}-\ln c_1\end{aligned}\right. \tag{8}$$

Equations (7) correspond to the expressions describing the resistance law in the boundary planetary layer of the atmosphere (cf [1, 3, 4]). However, in our case (in a shallow sea), unlike in the neutrally stratified planetary layer of the atmosphere, A and B are certain functions of non-dimensional parameters $\tau_x^a/\rho G^2$, $\tau_y^a/\rho G^2$, R_0, H/z_0. Let us find the form of these functions.

It appears from formulas (8) that A and B may be considered to be known when the distribution τ_x/ρ, τ_y/ρ in the layer $h\leqslant z\leqslant H$ is known, and also the values of $\frac{\partial\tau_x/\rho}{\partial z}$, $\frac{\partial\tau_y/\rho}{\partial z}$ at the level $z=h$. In order to determine τ_x/ρ, τ_y/ρ as functions of z, we differentiate equations (1) with respect to z and, taking into account the following expressions:

$$\tau/\rho=\tau_x/\rho+i\tau_y/\rho=k\frac{\partial u}{\partial z}+ik\frac{\partial v}{\partial z}, \tag{9}$$

as a result we obtain equation:

$$\frac{\partial^2\tau/\rho}{\partial z^2}+i\frac{f}{k}\,\tau/\rho=0. \tag{10}$$

We solve the above equation with the following boundary conditions:

$$\left.\begin{array}{ll}\tau/\rho=\tau^a/\rho & \text{at } z=H,\\ \tau/\rho=\tau^0/\rho & \text{at } z=h.\end{array}\right\} \tag{11}$$

The solution of problem (10), (11) has the following form:

$$\frac{\tau}{\rho}=-\frac{\tau^a/\rho\,\mathrm{sh}(1+i)\beta(h-z)-\tau^0/\rho\,\mathrm{sh}(1+i)\beta(H-z)}{\mathrm{sh}(1+i)\beta(H-h)},$$

where $\beta=\sqrt{f/2k}$.
Next we obtain:

$$\left.\frac{\partial\tau/\rho}{\partial z}\right|_{z=h}=\frac{(1+i)\beta}{\mathrm{sh}(1+i)\beta(H-h)}[\tau^a/\rho-\tau^0/\rho\,\mathrm{ch}(1+i)\beta(H-h)]. \tag{12}$$

We now take into consideration that for a shallow sea, inequalities $h\ll H$ and $\beta H<1$ are true. Thus, replacing the hyperbolic functions in (12) by their series with respect to argument βH and limiting ourselves to the first terms, we obtain:

$$\left.\frac{\partial\tau/\rho}{\partial z}\right|_{z=h}=\frac{1}{H}[(\tau_x^a/\rho-u_*^2)+i\tau_y^a/\rho], \tag{13}$$

where the frictional velocity is determined by $u_*=\sqrt{\tau^0/\rho}$. Separating in (13) the real part from the imaginary one and replacing them in (8), we have:

$$\left.\begin{array}{l}A=-\kappa\dfrac{u_*}{G}[\tau_x^a/\rho G^2\mathrm{R}_0(H/z_0)^{-1}(u_*/G)^{-2}-\mathrm{R}_0(H/z_0)^{-1}],\\ B=\kappa(u_*/G)^{-1}\tau_y^a/\rho G^2\mathrm{R}_0(H/z_0)^{-1}-\ln c_1.\end{array}\right\} \tag{14}$$

Equations (7) and (14) determine the resistance law in a shallow sea. Of course, this law holds only for a sea whose depth is limited by the condition $\beta H<1$, which – after substituting expressions (3), (6) – may be presented as:

$$H<\frac{u_*}{f}\sqrt{2c_1\kappa}. \tag{15}$$

It should be emphasized that depth H cannot be too small because in such a case the effect of the Coriolis force vanishes and equations (1) describe the vertical velocity profile incorrectly.

Let us consider a special case of the resistance law (7) and (14), when there is no wind above the free surface of the sea. In this case, the defining parameters do not include the terms τ_x^a/ρ, τ_y^a/ρ and the expressions for functions A and B take the form:

$$A=\kappa\frac{u_*}{G}\mathrm{R}_0(H/z_0)^{-1};\quad B=-\ln c_1. \tag{16}$$

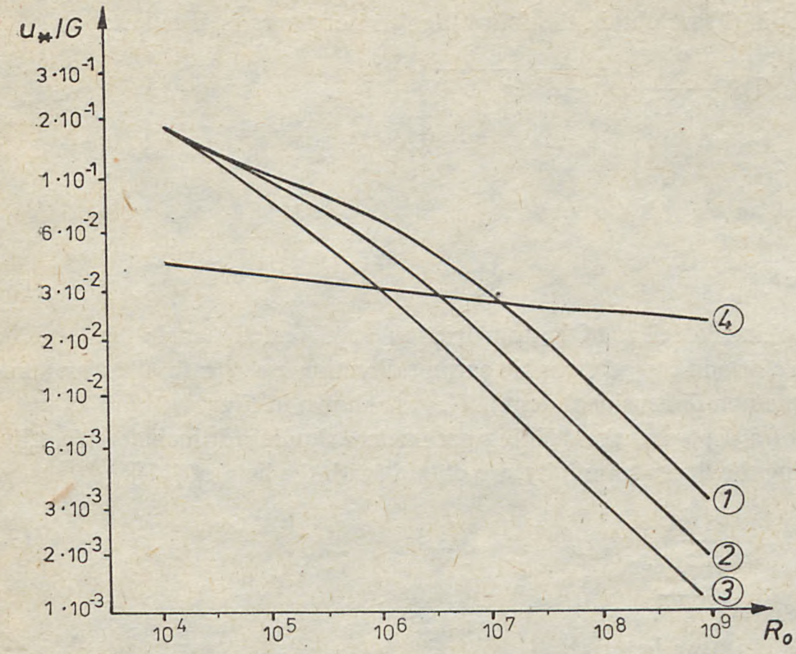

Fig. 1. Geostrophic coefficient of bottom friction (u_*/G) as a function of Rossby number (R_0) for different values of parameter H/z_0; $c_1=5\cdot10^{-3}$
1 $-H/z_0=1\cdot10^3$, *2* $-H/z_0=3\cdot10^3$; *3* $-H/z_0=1\cdot10^4$; *4* $-H\to\infty$

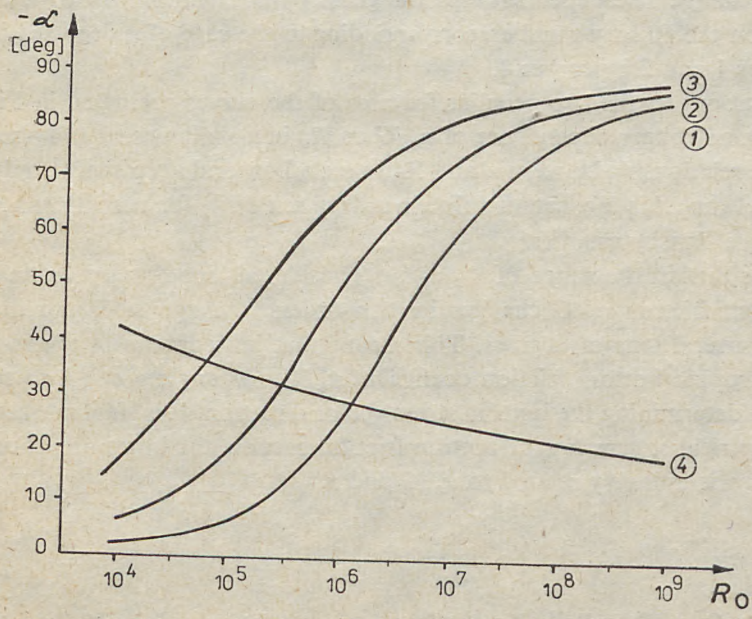

Fig. 2. Rotation angle α as a function of Rossby number (R_0) for different values of parameter H/z_0; $c_1=5\cdot10^{-3}$ Explanations – see Figure 1

Substituting (16) in (7) we obtain the following form of the resistance law:

$$\left.\begin{aligned}&\ln \mathrm{R}_0 = B - \ln(u_*/G) + \frac{1}{(u_*/G)}\sqrt{\kappa^2 - A'^2(u_*/G)^4}\\&\sin\alpha = -\frac{A'}{\kappa}(u_*/G)^2\,,\end{aligned}\right\} \tag{17}$$

where:

$$A' = \kappa \mathrm{R}_0 (H/z_0)^{-1}. \tag{18}$$

The geostrophic coefficient of bottom friction u_*/G and the angle between the bottom stress vector and the geostrophic current direction α *versus* the Rossby number and *versus* the non-dimensional depth H/z_0 are shown in Figures 1 and 2.

In the case of a deep sea, the defining parameters should not include τ_x^a/ρ, τ_y^a/ρ, and H. Increasing the depth to infinity, we obtain (with the help of (12)):

$$\left.\frac{\partial \tau_x/\rho}{\partial z}\right|_{z=h} = -\beta u_*^2; \qquad \left.\frac{\partial \tau_y/\rho}{\partial z}\right|_{z=h} = -\beta u_*^2, \tag{19}$$

where, as before, $\beta = \sqrt{f/2k}$.
Substituting (19) in (8) we have:

$$A = \sqrt{\kappa/2c_1}; \qquad B = -\sqrt{\kappa/2c_1} - \ln c_1\,. \tag{20}$$

Expressions (7) and (20) define the resistance law in the case of a deep sea. Thus, similarly to the planetary layer of the atmosphere with neutral stratification, A and B are universal constants. The dependence of the geostrophic coefficient of friction u_*/G and angle α on the Rossby number, corresponding to the case of a deep sea is presented in Figures 1 and 2 by curves *4*.

One should point out to the two peculiar features of the curves presented in Figure 1 and 2. Firstly, a stronger dependence of u_*/G on R_0 in a shallow sea, and – secondly – the reverse relationship between α and R_0 in a shallow and deep sea. It only results from the fact that A is a constant value in a deep sea while for a shallow sea it is a function of the Rossby number.

Let us also note that in the range of R_0 values considered here the geostrophic coefficient of bottom friction u_*/G changes by two orders of magnitude, and the veering angle – by several tens of degrees. This means that inaccuracies in the real values of the geostrophic bottom friction coefficient u_*/G and angle α can lead to significant errors in determining the bottom stress vector direction. It is similar when one applies the empirical square law of bottom friction because the bottom friction coefficient in this law is uniquely related to u_*/G and α.

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