R O Z P R A W Y

PAPERS

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ON A SYSTEM OF HYDRODYNAMIC EQUATIONS FOR CERTAIN OCEANOGRAPHICAL PROBLEMS IN THE REGION OF THE EARTH'S POLE AND THE STABILITY OF ITS SOLUTION

Abstract. In this paper, hydrodynamic equations are determined in a Cartesian co-ordinate system (the plane (x, y) of which passes through the parallel of latitude φ_o), by the so-called stereographic projective transformation. This system of hydrodynamic equations permits us to find its solution in the region of the earth's pole.

Also, a finite-difference method for the problem is described and the stability conditions of this scheme are discussed.

1. BASIC EQUATIONS IN SPHERICAL CO-ORDINATES

Let a be the radius of the Earth, D the depth of the basin. We assume that D is small as compared with a. The position of any point on the spherical sheet is specified by the angular coordinates φ, λ, where φ — latitude and λ — longitude. Let V_λ be the component of velocity along the

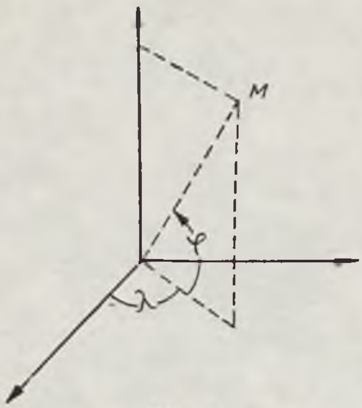

Fig. 1.
Ryc. 1.

parallel of latitude in the direction of increasing λ, V_φ the component along the meridian, in the direction of increasing φ (see Fig. 1). Also, let ζ denote the elevation of the free surface above the undisturbed level, A — horizontal eddy viscosity coefficient.

Furthermore, assuming that the components of velocity v_λ, are the values averaged from the sea floor to the surface and that the density is constant, one can write the following equations of motion and continuity in spherical co-ordinates.

$$\frac{\partial V_\lambda}{\partial t} + \frac{r\sqrt{V_\lambda^2 + V_\varphi^2}}{D+\zeta} V_\lambda - A\,\Delta_s V_\lambda - 2\omega \sin\varphi\, V_\varphi + \frac{g}{a\cos\varphi}\,\frac{\partial\zeta}{\partial\lambda} = F_\lambda \quad (1.1)$$

$$\frac{\partial V_\varphi}{\partial t} + \frac{r\sqrt{V_\lambda^2 + V_\varphi^2}}{D+\zeta} V_\varphi - A\,\Delta_s V_\varphi + 2\omega \sin\varphi\, V_\lambda + \frac{g}{a}\,\frac{\partial\zeta}{\partial\varphi} = F_\varphi \quad (1.2)$$

$$\frac{\partial\zeta}{\partial t} = \overline{V}\,\frac{1}{a\cos\varphi}\left\{\frac{\partial}{\partial\lambda}\left[(D+\zeta)\,v_\lambda\right] + \frac{\partial}{\partial\varphi}\left[(D+\zeta)\,v_\varphi\cos\varphi\right]\right\} \quad (1.3)$$

Where:

$$\Delta_s\Phi = \frac{1}{a^2}\left[\frac{1}{\cos\varphi}\,\frac{\partial}{\partial\varphi}\left(\cos\varphi\,\frac{\partial\Phi}{\partial\varphi}\right) + \frac{1}{\cos^2\varphi}\,\frac{\partial^2\Phi}{\partial\lambda^2}\right] \quad (1.4)$$

is Laplace's operator in spherical co-ordinates; Φ is one of the above two functions V_λ, V_φ; F_λ, F_φ are components of extraneous forces in the λ-direction and φ-direction, respectively; r — bottom-stress coefficient having the constant dimensionless value $r = 3 \cdot 10^{-3}$. This value depends on the roughness of the bottom and the bottom material. In the deep seas the variation of the value of r with respect to the depth is small. The above value seems to be applicable in the problems of estuaries and open seas as well as in the ocean (Dronkers, 1964).

It is evident that the system of equations (1.1) — (1.3) can be integrated only in an area not containing the earth's pole. Thus, for the hydrodynamic problems in the region of the earth's pole, it is necessary to write a new system of hydrodynamic equations, the solution of which can be determined in this region. To achieve this aim a stereographic projective transformation will be introduced in this study.

2. STEREOGRAPHIC PROJECTION AND STEREOGRAPHIC POLAR SYSTEM OF CO-ORDINATES

Let us take a plane Q passing through the parallel of latitude φ_0. If we connect an arbitrary point of the earth's surface to the South pole of the earth (considered as the centre of projection), then there is a one-to-one correspondence between the point M and M' of the earth's surface and of the plane **Q** (see Fig. 2).

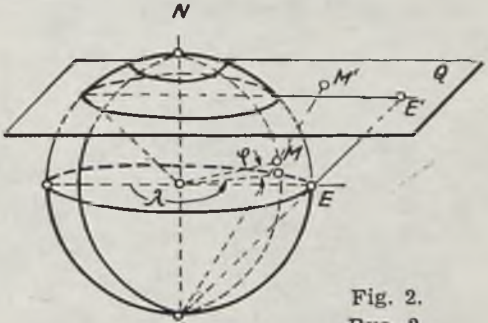

Fig. 2.
Ryc. 2.

This projection is called stereographic and the plane **Q** is called the stereographic projective plane. By this transformation the parallels of latitude of the earth are transformed into concentric circles on the plane Q (see Fig. 3), and the meridians — into half lines with initial point P, which is the projection of the earth's pole on the plane (Rektorys, 1969).

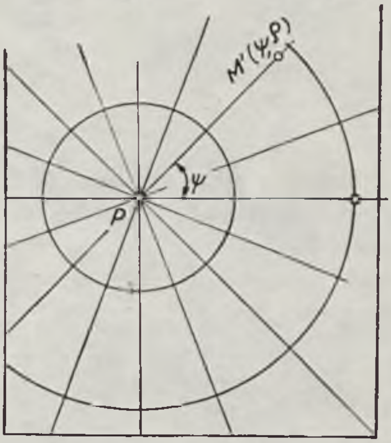

Fig. 3.
Ryc. 3.

Let us introduce into the plane Q a polar co-ordinate system, whose origin P coincides with the stereographic projection of the earth's pole and whose polar semi-axis p coincides with the stereographic projection of the originative meridian (Greenwich). In this co-ordinate system, the position of an arbitrary point M' may be determined by the polar coordinates ϱ, ψ; here ϱ is the distance of the point M' from the origin P, ψ is the directed angle between the segment PM' and the polar semi-axis. In order to establish a one-to-one correspondence between the points of the plane and the pairs of numbers (ϱ, ψ), it is necessary to restrict the co-ordinates ϱ and ψ in the following way:

$$\varrho \geqslant 0, \quad 0 \leqslant \psi < 2\pi$$

Throughout this paper the polar co-ordinate system introduced above is called, for convenience sake, the stereographic polar system of co-ordinates (ϱ, ψ).

3. TRANSFORMATION OF HYDRODYNAMIC EQUATIONS IN SPHERICAL CO-ORDINATES INTO STEREOGRAPHIC POLAR CO-ORDINATES

Let us denote by K the distance between points P and E' (Fig. 4). From the transformation mentioned above, the following relations may be determined. First it is evident that:

$$\lambda = \Psi \tag{3.1}$$

$$\frac{\partial}{\partial \lambda} = \frac{\partial}{\partial \Psi} \tag{3.2}$$

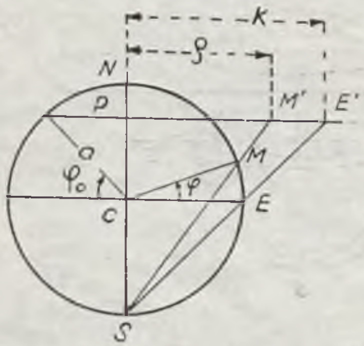

Fig. 4.
Ryc. 4.

(see Figs. 2 and 3).

Then, from the isosceles triangle SPE' and SPM' (see Fig. 4), it follows:

$$K = PE' = PS = PC + CS$$

$$K = 2a \cos^2 \left(45° - \frac{\varphi_0}{2}\right) \tag{3.3}$$

and

$$\frac{\varrho}{K} = \tan \left(\frac{90° - \varphi_0}{2}\right) \tag{3.4}$$

Introducing a new variable ϱ, the partial derivative with respect to the spherical coordinate φ can be written as:

$$\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial \varrho} \cdot \frac{\partial \varrho}{\partial \varphi}$$

$$\frac{\partial}{\partial \varphi} = \left(-\frac{1}{a}\frac{\partial \varrho}{\partial \varphi}\right)\left(-a\frac{\partial}{\partial \varrho}\right) \tag{3.5}$$

By substitutions

$$m = -\frac{1}{a}\frac{\partial \varrho}{\partial \varphi}$$

one has:

$$\partial \varphi = -\frac{1}{ma}\partial \varrho \tag{3.6}$$

and

$$\frac{\partial}{\partial \varphi} = -ma\frac{\partial}{\partial \varrho} \tag{3.7}$$

From the relations (3.4) — (3.5) the following formulae can be determined:

$$m = \frac{K\left[1 + \left(\frac{\varrho}{K}\right)^2\right]}{2a} \tag{3.8}$$

and

$$m = \frac{K}{a(1+\sin\varphi)} \tag{3.9}$$

It should be noticed that, φ is the latitude of the point M' on the stereographic projection plane and may be deduced from equalities (3.8) — (3.9) as:

$$\sin\varphi = \frac{1-\left(\frac{\varrho}{K}\right)^2}{1+\left(\frac{\varrho}{K}\right)^2} \tag{3.10}$$

Here ϱ is the distance of the point M' from the origin P of the stereographic polar co-ordinate system.

By application of the transformations (3.2), (3.7) and relations (3.9) — (3.10), the terms of equations (1.1) — (1.3) in spheric co-ordinates can be expressed through stereographic polar co-ordinates (Belov, 1967).

For velocity components in spheric co-ordinates, we have relations:

$$V_\lambda = a\cos\varphi\,\lambda \tag{3.11}$$

$$V_\varphi = a\dot{\varphi} \tag{3.12}$$

The substitution of (3.2), (3.6), (3.7), (3.9) and (3.10) by (3.11), (3.12) gives

$$V_\lambda = \frac{1}{m}(\varrho\dot{\psi}) \tag{3.13}$$

$$V_\varphi = -\frac{1}{m}(\dot{\varrho}) \tag{3.14}$$

It is easily seen that the expressions in parentheses are the velocity components V_ψ, V_ϱ in the stereographic polar co-ordinates introduced above in the direction ψ and ϱ, respectively. Thus, relations (3.13), (3.14) can be rewritten as

$$V_\lambda = \frac{1}{m}V_\psi \tag{3.15}$$

$$V_\varphi = -\frac{1}{m}V_\varrho \tag{3.16}$$

Let Δ_p denote Laplace's operator in the polar co-ordinate system (ϱ, ψ). By taking into account (1.4) with expressions (3.2), (3.7), (3.10) and considering relations (3.15), (3.16) we have:

$$\Delta_s V_\lambda = \frac{1}{a^2}\left\{\frac{ma}{\varrho}\,ma\,\frac{\partial}{\partial\varrho}\left[\frac{\varrho}{ma}\,ma\,\frac{\partial\left(\frac{1}{m}V_\psi\right)}{\partial\varrho}\right] + \frac{m^2a^2}{\varrho^2}\,\frac{\partial^2\left(\frac{1}{m}V_\psi\right)}{\partial\psi^2} = \right.$$

$$= m^2 \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial \left(\frac{1}{m} V_\psi \right)}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \left(\frac{1}{m} V_\psi \right)}{\partial \psi^2} \right] \Bigg\}$$

It is easily seen that the expression in brackets is Laplace's operator in the polar co-ordinates system mentioned above.

Thus,

$$\Delta_s V_\lambda = m^2 \Delta_p \left(\frac{1}{m} V_\psi \right) \tag{3.17}$$

Similarly, we have:

$$\Delta_s V_\varphi = m^2 \Delta_p \left(- \frac{1}{m} V_\varrho \right) \tag{3.18}$$

By introducing (3.17) — (3.18) into equations (1.1) — (1.3) and carrying out relations (3.2), (3.7), (3.10), (3.15), (3.16) with the functions and their first-order derivatives in the equations and calling components of extraneous force in the ψ — direction and ϱ — direction by F_ψ, F_ϱ, respectively, we obtain the system of hydrodynamic equations in stereographic polar co-ordinates:

$$\frac{\partial}{\partial t} \left(\frac{V_\psi}{m} \right) + \frac{r \frac{V_\psi}{m} \sqrt{\frac{V_\psi^2 + V_\varrho^2}{m^2}}}{D + \zeta} - m^2 A \Delta_p \left(\frac{V_\psi}{m} \right) + 2\omega \sin \varphi \left(\frac{V_\varrho}{m} \right) + mg \frac{\partial \zeta}{\partial \psi} =$$
$$= F_\varphi \tag{3.19}$$

$$\frac{\partial}{\partial t} \left(\frac{V_\varrho}{m} \right) + \frac{r \frac{V_\varrho}{m} \sqrt{\frac{V_\psi^2 + V_\varrho^2}{m^2}}}{D + \zeta} - m^2 A \Delta_p \left(- \frac{V_\varrho}{m} \right) - 2\omega \sin \varphi \left(\frac{V_\psi}{m} \right) -$$
$$- mg \frac{\partial \zeta}{\partial \varrho} = F_\varrho \tag{3.20}$$

$$\frac{\partial \zeta}{\partial t} + \frac{m}{\varrho} \left\{ \frac{\partial}{\partial \psi} \left[(D + \zeta) \frac{V_\psi}{m} \right] - \varrho \frac{\partial}{\partial \varrho} \left[(D + \zeta) \frac{V_\varrho}{m} \right] \right\} = 0 \tag{3.21}$$

where, the value of $\sin \varphi$ is determined by relation (3.10).

Since the system of equations (3.19) — (3.21) cannot be integrated at the earth's pole, where $\varrho = 0$, we will write them in the Cartesian co-ordinates.

4. HYDRODYNAMIC EQUATIONS IN THE CARTESIAN CO-ORDINATE SYSTEM SITUATED ON THE STEREOGRAPHIC PROJECTIVE PLANE

The system of equations (3.19) — (3.21) might be turned into an arbitrary Cartesian co-ordinate system, whose origin will not coincide with stereographic polar origin P and the x-axis of which makes an angle with the polar semi—axis p. Howerer, this transformation would require much effort and the new system of equations would be fairly complicated. Hence, it is more advantageous to introduce into the stereographic projective plane Q the Cartesian co-ordinate system (0; x, y), whose origin is arbitrary, x — axis is parallel to the polar semi—axis which is similarly oriented.

If x_o, y_o denote co-ordinates of the origin P in the system (Q; x, y) and x, y — co-ordinates of an arbitrary point M on the projective plane Q, then stereographic polar co-ordinates and Cartesian co-ordinates are connected by relations

$$x = \varrho \cos \psi + x_o \tag{4.1}$$

$$y = \varrho \sin \psi + y_o \tag{4.2}$$

Also, let F_x, F_y denote the components of extraneous forces in the x — direction, y — direction, respectively, of the Cartesian co-ordinate system. The following relations are deduced:

$$F_\psi = - \sin \psi \, F_x + \cos \psi \, F_y \tag{4.3}$$

$$F_\varrho = - \cos \psi \, F_x - \sin \psi \, F_y \tag{4.4}$$

On the other hand, from relations (4.1), (4.2) and formulae

$$V_\psi = \varrho \psi'$$

$$V_\varrho = \dot{\varrho}$$

we can deduce:

$$V_\psi = - \sin \psi \, \dot{x} + \cos \psi \, \dot{y} \tag{4.5}$$

$$V_\sigma = \cos \psi \, \dot{x} + \sin \psi \, \dot{y} \tag{4.6}$$

By introducing relations (4.3) — (4.6) into the equations (3.19) — (3.21) we obtain the system of hydrodynamic equations in the Cartesian co-ordinate system mentioned as:

$$\frac{\partial u}{\partial t} - m^2 A \Delta u + \frac{r}{D} v \sqrt{u^2 + v^2} + 2 \omega u \sin \varphi + mg \frac{\partial \zeta}{\partial x} = F_x \tag{4.7}$$

$$\frac{\partial v}{\partial t} - m^2 A \Delta v + \frac{r}{D} u \sqrt{u^2 + v^2} - 2 \omega v \sin \varphi + mg \frac{\partial \zeta}{\partial y} = F_y \tag{4.8}$$

$$\frac{\partial \zeta}{\partial t} + m \frac{\partial [(D + \zeta) u]}{\partial x} + m \frac{\partial [(D + \zeta) v]}{\partial y} \tag{4.9}$$

$$u = \frac{\dot{x}}{m}, \qquad v = \frac{\dot{y}}{m},$$

Δ — Laplace's operator in Cartesian co-ordinates.

It is evident that, equations (4.7) — (4.9) in Cartesian co-ordinates permit us to find their solution in a region containing the earth's pole. Thus the first aim of this paper is achieved.

Further considerations deal with finite difference schemes of these hydrodynamic equations, with and without the effect of eddy viscosity. Also, their stability conditions are discussed.

5. DIFFERENCE SCHEME WITHOUT THE EFFECT OF EDDY VISCOSITY

a) System of finite-difference equations

To obtain a digital solution to the above partial differential equations, it is necessary to write down finite difference forms of the equations on time and to choose a space grids. Central time differences are used for the velocity components u, v and forward time differences are used for the water levels. Also, central differences are used for all space derivatives. A space-staggered quadratic grid with mesh size $l = 2h$ is used, where velocity components and water level are described at different grid-points (see Fig. 5).

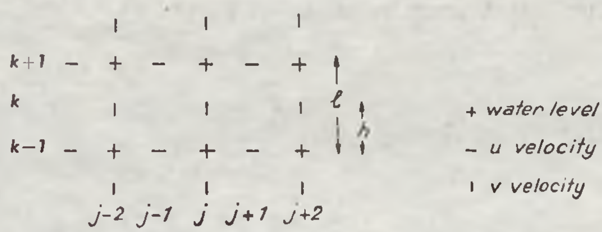

Fig. 5.
Ryc. 5.

We denote each function $\Phi(x, y, t)$ of $x = j\Delta x$, $y = k\Delta y$ and $t = n\Delta t$ by $\Phi^n_{j,k}$ where j, k are integers, $\Delta x = \Delta y = l = 2h$, $\Delta t = \tau$ is time step.

In connection with the finite-difference methods, some mathematical considerations concerning stability are unavoidable and the solution must be restricted to linearized systems. Additionally, the mean values of u, v are introduced since they appear in the terms of bottom stress and Coriolis force (Hansen, 1962):

$$\bar{u}_{j,k} = \frac{1}{4}(u_{j+1,k+1} + u_{j+1,k-1} + u_{j-1,k-1} + u_{j-1,k+1}) \tag{5.1}$$

$$\bar{v}_{j,k} = \frac{1}{4}(v_{j+1,k+1} + v_{j+1,k-1} + v_{j-1,k-1} + v_{j-1,k+1}) \tag{5.2}$$

Hence,

$$R_x = \frac{r}{D}\sqrt{u^2 + v^2}$$

$$R_y = \frac{r}{D}\sqrt{u^2 + v^2}$$

The finite difference scheme of system (4.15) — (4.17) may be written down as:

$$u^{n+1}_{j+1,k} = (1 - 2R_x\,\tau)\,u^{n-1}_{j+1,k} + \frac{f\tau}{2}\,\bar{v}^{n-1}_{j+1,k} - m\,\frac{g\tau}{h}\,(\zeta^{n}_{j+2,k} - \zeta^{n}_{j,k}) \qquad (5.3)$$

$$v^{n+1}_{j+1,k} = (1 - 2R_y\,\tau)\,v^{n-1}_{j,k+1} - \frac{f\tau}{2}\,\bar{u}^{n-1}_{j,k+1} - m\,\frac{g\tau}{h}\,(\zeta^{n}_{j,k+2} - \zeta^{n}_{j,k}) \qquad (5.4)$$

$$\zeta^{n+2}_{j,k} = \zeta^{n}_{j,k} - m\,\frac{\tau}{h}\Big\{[u(D+\zeta)]^{n+1}_{j+1,k} - [u(D+\zeta)]^{n+1}_{j-1,k} + [v(D+\zeta)]^{n+1}_{j,k+1} - [v(D+\zeta)]^{n+1}_{j,k-1}\Big\} \qquad (5.5)$$

b) Approximations of boundary conditions

In order to get the simplest statement of the boundary conditions, a rectangular area is considered the sides of which are parallel to the x-axis and y-axis and which consists of open and closed boundaries. Real basins rarely have rectilinear boundaries, but if h is small relative to the basin dimension, the true boundary can be approximated by a short section of a zigzag boundary. In this case the boundary conditions used are in the form of zero velocity components normal to the closed boundaries and at the open boundary the water levels are given as time functions (see Fig. 6).

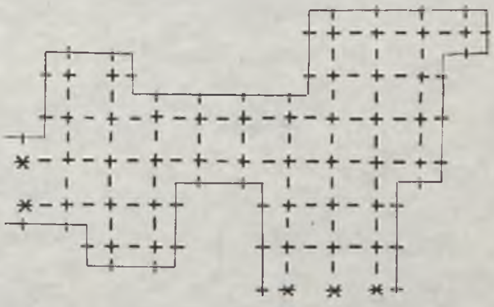

Fig. 6. Grid example
+ water level
— u velocity
| v velocity
* water level at open boundaries given as a function of time

Ryc. 6. Przykład siatki dla obliczeń numerycznych
+ zmiany poziomu wody | v prędkość
— u prędkość
*zmiany poziomu na otwartych granicach są dane jako funkcja czasu

c) Computational stability

In the analysis of the computational stability, the decay of waves over the combinet steps is analysed through limitations put on the amplification matrix from the time level n to the time level n + 1 (Richtmyer, 1967). This may be done by using the general Fourier type solution.

$$
\begin{aligned}
\zeta^{n}_{j,k} &= \zeta^{n} \exp\,[i(\alpha l + \beta l)] \\
u^{n}_{j,k} &= u^{n} \exp\,[i(\alpha l + \beta l)] \\
v^{n}_{j,k} &= v^{n} \exp\,[i(\alpha l + \beta l)]
\end{aligned}
\tag{5.6}
$$

Introducing these relations into equations (5.3) — (5.5) and assuming that $R_x = R_y = R =$ const., $D \gg \zeta$ and $D =$ const. we obtain:

$$u^{n+1} = au^{n-1} + b\,v^{n-1} - 2i\,m\frac{g\tau}{h}\sin(\alpha l)\zeta^{n} \tag{5.7}$$

$$v^{n+1} = a\,v^{n-1} - b\,u^{n-1} - 2i\,m\frac{g\tau}{h}\sin(\beta l)\zeta^{n} \tag{5.8}$$

$$\zeta^{n+2} = \zeta^{n} - m\frac{\tau}{h}D[2i\sin(\alpha l)u^{n+1} + 2i\sin(\beta l)\;v^{n+1}] \tag{5.9}$$

where:

$$a = 1 - 2R\tau \tag{5.10}$$

$$b = 2f\tau\cos(\alpha l)\cos(\beta l) \tag{5.11}$$

By eliminating u^{n+1} and v^{n+1} from (5.9) with the use of (5.7), (5.8) and introducing

$$
\begin{aligned}
u^{n+1}_{1} &= u^{n} \\
v^{n+1}_{1} &= v^{n} \\
\zeta^{n+1} &= \zeta^{n}_{1}
\end{aligned}
$$

we have:

$$
\begin{bmatrix} u^{n+1} \\ u^{n+1}_{1} \\ v^{n+1} \\ v^{n+1}_{1} \\ \zeta^{n+1} \\ \zeta^{n+1}_{1} \end{bmatrix}
= G
\begin{bmatrix} u^{n} \\ u_1^{n} \\ v^{n} \\ v_1^{n} \\ \zeta^{n} \\ \zeta_1^{n} \end{bmatrix}
\tag{5.12}
$$

Where:

$$G = \begin{bmatrix} 0 & a & 0 & b & -2im\frac{g\tau}{h}\sin\alpha l & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b & 0 & a & -2im\frac{g\tau}{h}\sin\beta l & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2im\frac{\tau D}{h}d_1 & 0 & -2im\frac{\tau D}{h}d_2 & d_3 & 0 \end{bmatrix} \quad (5.13)$$

$$d_1 = a\sin\alpha l - b\sin\beta l$$

$$d_2 = a\sin\beta l + b\sin\alpha l$$

$$d_3 = 1 - 4m^2\frac{\tau^2}{h^2} g D(\sin^2\alpha l + \sin^2\beta l) \quad (5.14)$$

The von Neumann stability condition requires that the eigenvalues of the amplification matrix G not exceed unity in absolute value, for physically stable systems. The eigenvalues of the matrix G can be obtained from the characteristic equation:

$$|G - \lambda E| = 0 \quad (5.15)$$

This equation may be expanded into

$$(\lambda^2 - 1)[\lambda^4 - 2a\lambda^2 + (a^2 + b^2)] + 4m^2\frac{\tau^2}{h^2}gD(\sin^2\alpha l + \sin^2\beta l)\lambda^2(\lambda^2 - a) = 0 \quad (5.16)$$

Solutions of equation (5.16) depend on the values of bottom stress and Coriolis terms, hence these solutions must be considered for different values of these terms

Case 1. $R = f = 0$

Where the effect of bottom stress and Coriolis force is omitted, we have $a = 1$, $b = 0$.

Equation (5.16) reduces to

$$(\lambda^2 - 1)\left\{\lambda^4 - 2\lambda^2\left[1 - 2m^2\frac{\tau^2}{h^2}gD(\sin^2\alpha l + \sin^2\beta l)\right] + 1\right\} = 0 \quad (5.17)$$

The solutions of this equation are:

$$\lambda_{1,2} = \pm 1$$

$$\lambda_{3,4} = 1 - 2m^2\frac{\tau^2}{h^2}gD(\sin^2\alpha l + \sin^2\beta l) \pm$$

$$\pm 2i\left\{m^2\frac{\tau^2}{h^2}gD(\sin^2\alpha l + \sin^2\beta l)\left[1 - m^2\frac{\tau^2}{h^2}gD(\sin^2\alpha l + \sin^2\beta l)\right]\right\}^{1/2} \quad (5.18)$$

From (5.18) it follows that $|\lambda_{3,4}| = 1$ for

$$m^2 \frac{\tau^2}{h^2} gD(\sin^2 \alpha l + \sin^2 \beta l) < 1 \tag{5.19}$$

Because $\sin^2 \alpha l \leqslant 1$, $\sin^2 \beta l \leqslant 1$, the following stability criterium can be deduced from (5.19).

$$\tau \leqslant \frac{h}{m \sqrt{2g D}} \tag{5.20}$$

It is seen that this criterium is different from that shown by Harris, Jeleśniański (1964) and Kagan (1970) by a factor $\frac{1}{m}$.

Case 2. in which $R \neq 0, f \neq 0$

In the general case, the highest absolute value corresponds to $L = 2\,l = 4h$, the shortest wave-length which may be resolved the grid.

Thus, we have $\alpha l = \beta l = \frac{2\pi}{L} l = \pi$ and we obtain from (5.10)

$$a = 1 - 2R\,\tau, \quad b = 2f\,\tau \tag{5.23}$$

Then equation (5.17) reduces to

$$(\lambda^2 - 1)[\lambda^4 - 2(1 - 2R\,\tau)\lambda^2 + (1 - 2R\,\tau)^2 + 4f^2\tau^2] = 0 \tag{5.24}$$

Roots of (5.24) are:

$$\lambda_{1,2} = \pm 1$$
$$\lambda^2_{3,4} = 1 - 2R\,\tau \pm 2i\,f\,\tau$$
$$|\lambda^2_{3,4}| = 1 - 4\tau[R - (R^2 + f^2)\tau]$$

We wish $|\lambda_{3,4}|$ to be equal or less than unity; hence we must have

$$1 - 4\tau[R - (R^2 + f^2)\tau] \leqslant 1 \tag{5.25}$$

Solving this quadratic inequality we obtain stability condition:

$$\tau \leqslant \frac{R}{R^2 + f^2} \tag{5.26}$$

From this the following special cases can be obtained:

If $f = 0$ we have

$$\tau \leqslant \frac{1}{R}$$

and if $R \ll f$ we have:

$$\tau \leqslant \frac{R}{f^2}$$

6. DIFFERENCE SCHEME WITH THE EFFECT OF EDDY VISCOSITY

Where the effect of eddy viscosity is considered, investigations are conducted in a similar manner to that described in section 5. All notations and assumptions used above hold. Thus the system of finite difference equations may be written down as:

$$u^{n+1}_{j+1,k} = (1 - 2R_x \tau)\, u^{n-1}_{j+1,k} + \frac{f\tau}{2}\left(v^{n-1}_{j,k+1} + v^{n-1}_{j,k-1} + v^{n-1}_{j+2,k-1} + v^{n-1}_{j+2,k+1}\right)$$
$$+ \frac{m^2 A\tau}{2h^2}\left(u^{n-1}_{j+3,k} + u^{n-1}_{j-1,k} + u^{n-1}_{j+1,k+2} + u^{n-1}_{j+1,k-2} - 4u^{n-1}_{j+1,k}\right)$$
$$- \frac{mg\tau}{h}\left(\zeta^{n}_{j+2,k} - \zeta^{n}_{j,k}\right) \tag{6.1}$$

$$v^{n+1}_{j,k+1} = (1 - 2R_y \tau)\, v^{n-1}_{j,k+1} + \frac{f\tau}{2}\left(u^{n-1}_{j-1,k} + u^{n-1}_{j+1,k} + u^{n-1}_{j-1,k+2} + u^{n-1}_{j-1,k+2}\right)$$
$$+ \frac{m^2 A\tau}{2h^2}\left(v^{n-1}_{j-1,k+1} + v^{n-1}_{j+2,k+1} + v^{n-1}_{j,k-1} + v^{n-1}_{j,k+3} - 4v^{n-1}_{j,k+1}\right)$$
$$- \frac{mg\tau}{h}\left(\zeta^{n}_{j,k+2} - \zeta^{n}_{j,k}\right) \tag{6.2}$$

$$\zeta^{n+2}_{j,k} = \zeta^{n}_{j,k} - m\frac{\tau}{h}\left\{[u(D+\zeta)]^{n+1}_{j+1,k} - [u(D+\zeta)]^{n+1}_{j-1,k}\right.$$
$$\left. + [v(D+\zeta)]^{n+1}_{j,k+1} - [v(D+\zeta)]^{n+1}_{j,k+1}\right\} \tag{6.3}$$

First, the general case with bottom-stress and Coriolis terms is discussed. In this case the amplification matrix of the system of difference equations (6.1) — (6.3) is:

$$G' = \begin{bmatrix} 0 & a' & 0 & b & -2\,im\frac{g\tau}{h}\sin\alpha l & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b & 0 & a' & -2\,im\frac{g\tau}{h}\sin\beta l & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2im\frac{\tau D}{h}d_1' & 0 & -2im\frac{\tau D}{h}d_2' & d_3 & 0 \end{bmatrix} \tag{6.4}$$

where:

$$a' = 1 - 2R\tau - 2m^2\frac{A\tau}{h^2}(\sin^2\alpha l + \sin^2\beta l) \tag{6.5}$$

$$d_1' = a' \sin\alpha l - b \sin\beta l$$
$$d_2' = a' \sin\beta l - b \sin\alpha l \tag{6.6}$$

Characteristic equation of matrix G' is

$$|G' - \lambda E| = o \tag{6.7}$$

Expanding equation (6.7) with the use of relations (6.6), we obtain:

$$(\lambda^2-1)\,[\lambda^4-2a'\lambda^2+(a'^2+b^2)] + 4\,m^2\,\frac{\tau^2}{h^2}\,gD\,(\sin^2\alpha l+\sin^2\beta l)\,\lambda^2\,(\lambda^2-a') = o \tag{6.8}$$

Restricting to the case of the largest roots of the characteristic equation (6.8) we have:

$$a' = 1-2R\tau$$
$$b = 2f\tau$$

Then equation (6.8) becomes:

$$(\lambda^2-1)\,[\lambda^4-2\,(1-2R\tau)\,\lambda^2+(1-2R\tau)^2+4f^2\tau^2] = o \tag{6.9}$$

The roots of equation (6.9) are:

$$\lambda_{1,2} = \pm 1$$
$$\lambda^2_{3,4} = 1 - 2R\tau \pm 2if\tau$$

and absolute values of eigen values are

$$|\lambda_{1,2}| = 1$$
$$|\lambda^2_{3,4}|^2 = 1-4\,[R\tau - (R^2+f^2)\,\tau^2] \tag{6.10}$$

To keep the roots $\lambda_{3,4}$ in or on the unit circle, we must have

$$1 - 4\,[R\tau - (R^2+f^2)\,\tau^2] \leqslant 1$$ or

$$\tau \leqslant \frac{R}{R^2+f^2} \tag{6.11}$$

from the stability conditions (5.26) and (6.11) it is seen that the effects of eddy viscosity have no influence on the amplitude of waves with the shortest wave-length.

In all other cases this effect generates the decay of wave amplitude. Consequently, conditions (5.20), (6.11) are sufficient to maintain the stability of the difference scheme (6.1) — (6.2).

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UKŁAD RÓWNAŃ HYDRODYNAMICZNYCH OPISUJĄCYCH ZJAWISKA OCEANOGRAFICZNE W REJONIE BIEGUNÓW ORAZ NUMERYCZNA STABILNOŚĆ UKŁADU

Streszczenie

Rozpatrzono układ równań ruchu i ciągłości, opisujący fale długie na sferze (1.1, 1.2, 1.3). Jednak, jak wiadomo, równania we współrzędnych sferycznych nie posiadają rozwiązania na biegunie, kiedy $\cos\varphi = 0$. W pracy zaproponowano wprowadzenie dodatkowego układu współrzędnych Descarta znajdujących się na płaszczyźnie Q (ryc. 2).

Przekształcenie rejonu podbiegunowego z sfery na płaszczyznę Q dokonuje się za pomocą stereograficznej transformacji współrzędnych. W rezultacie otrzymano układ równań (4.7, 4.8, 4.9).

Ponieważ rozwiązanie konkretnych problemów oceanograficznych może być wykonane tylko za pomocą równań różnicowych, wprowadzono siatkę obliczeń numerycznych (ryc. 5), a następnie układ równań różnicowych (5.3, 5.4, 5.5).

Stabilność numeryczna tego układu została przedyskutowana przez zastosowanie macierzy przejścia G — wyrażenie (5.12), od kroku czasowego n do n + 1. Wartości własne (λ) tej macierzy dla stabilności układu równań winny spełniać nierówność $|\lambda| \leqslant 1$. W rezultacie otrzymano kryterium stabilności wiążące krok czasowy (τ) oraz krok siatki przestrzennej (h)

$$\tau \leqslant \frac{h}{m\sqrt{2gD}} \tag{5.20}$$

gdzie

g — przyśpieszenie ziemskie
D — głębokość oceanu
m — współczynnik przejścia od współrzędnych sferycznych do współrzędnych Descartesa,

oraz zmodyfikowane kryterium, gdy występują siła Coriolisa i siła tarcia przydennego

$$\tau \leqslant \frac{R}{R^2 + f^2} \tag{5.26}$$

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